WAVE PROPAGATION
OVER NON-UNIFORM ELECTRICAL CONDUCTORS*

BY

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Introduction

The main object of the mathematical theory developed in this paper is the analysis of the propagation of electrical waves over a conductor represented in Fig. 3, § III, below. This conductor consists of a loop of wire $L_1 \cdots L_{k+1}$ of length $2l$ in which there are, at equidistant points, the so-called *reactance points*, a certain number of equal coils interposed dividing the loop into a number of equal parts, called the *interstices* of the loop. These coils may have condensers in series with them or they may have secondary circuits with condensers. The propagation of electrical waves over a periodically loaded loop of this kind is compared with that over a uniform loop having the same total inductance, resistance, and capacity. This uniform loop is called the *corresponding uniform conductor* of the periodically loaded loop.

A similar problem in mechanics is that of the forced and free vibrations of a periodically loaded heavy, flexible, inextensible string of finite length taking frictional resistances into account. To my knowledge, neither the electrical problem nor its corresponding mechanical problem has been investigated before.

It will be observed in the course of this paper that a study of the propagation of electrical waves over a periodically loaded conductor of this kind suggests forcibly an electromagnetic theory of emission and absorption of light by molecular complexes, which on account of the physical conception underlying it, if for no other reason, possesses many attractive features. In this theory the ether and the material ions imbedded in it correspond to the uniform wire and the reactance points considered in this paper. These matters, however, are of a more or less speculative character and have, therefore, no place here. But it should be noted that the physical problem discussed here was first suggested by speculations of this kind. It can be stated as follows: Under what conditions will the non-uniform conductor represented in Fig. 3 be approximately equivalent to its corresponding uniform conductor? Or, to be more precise: For what fre-

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quency will an electrical wave have approximately the same wave-length and the same damping or attenuation constant on one conductor as on the other?

The mathematical theory developed here gives a definite answer to this question. This answer can be stated in a few words and for that purpose it is desirable to introduce here a new technical term, the so-called *angular distance* of the interstices, that is, of the interval between consecutive reactance points. This interval is at any given frequency a definite fraction of the wave-length corresponding to that frequency. Let this fraction be $\phi/2\pi$; then ϕ is the angular distances of the interstices. The angular distance of a wave-length is, of course, 2π. The general rule expressing the conditions of equivalence of a non-uniform conductor to its corresponding uniform conductor can now be expressed as follows: *For any given frequency a non-uniform conductor of the second type is equivalent to its corresponding uniform conductor as nearly as* $\sin \phi/2$ *is to* $\phi/2$.

The higher the frequency the less resemblance will there be between a given non-uniform conductor and its corresponding uniform conductor. Conversely, if this resemblance is sufficiently close for a given frequency it will be closer for all lower frequencies. When the half wave-length under consideration becomes smaller than the interstices, then the resemblance, as far as that wave-length and all shorter wave-lengths are concerned, ceases altogether.

A brief summary of this paper will now be given.

The main object of this research is the solution of the problem of § III. This solution depends on the solution of the problem of § II, and this again is moulded after the pattern of the solution of the problem of § I.

In § I the wave propagation over a uniform wire conductor represented in Fig. 1 is discussed. The effect of the transmitting apparatus A and of the receiving apparatus B is taken into account. Equation (5) is the most general solution of this case. The propagation of waves of both forced and free periods is easily deduced from it. This particular form of the general solution is new and it was selected because it is best suited for comparing the propagation of waves over the non-uniform conductors of §§ II and III with that over their corresponding uniform conductors.

In § II the wave propagation over a non-uniform conductor of the first type represented in Fig. 2 is discussed. This conductor consists of a certain number of equal coils $L_1, \cdots, L_n$ which are connected in series. A certain number of equal condensers, one at each juncture between two consecutive coils, connect this conductor to ground. A transmitting apparatus A and a receiving apparatus B are present. The general solution of this problem, equation (6*a*), is moulded after the pattern of equation (5) of § I. Both forced and free oscillations on a conductor of this kind are considered, and the conditions under which it becomes equivalent to its corresponding uniform conductor are worked out at considerable length. The problem of this section and its solution are both new.

The problem of § III can, from a purely mathematical point of view, be stated as follows: Find the integral of the following partial differential equation:

$$L\frac{d^2y}{dt^2} + R\frac{dy}{dt} = \frac{1}{C}\frac{\partial^2 y}{\partial s^2},$$

and determine it in such a way as to satisfy $k + 2$ boundary conditions. Determine also the conditions under which this integral will be equivalent to the integral represented in equation (5), § I.

Equation (2) represents the most general solution, and the constants $x_1, x_2, \cdots, x_{k+2}$ have to be determined from the $k + 2$ boundary conditions. The principal mathematical difficulty here reduces itself then to the proper mathematical formulation of these boundary conditions so as to obtain a system of equations which can be readily solved. Such a system is system (3) of this section. It is of the same form as system (6) of § II, the solution of which was obtained in that section. Equations (4) and (5) are thus obtained. Equation (5) is the most general solution, and when this equation becomes very nearly the same as equation (5) of § I then the non-uniform conductor of this section becomes equivalent to its corresponding uniform conductor. This equivalence cannot be decided without a careful study of the wave-lengths and of the damping constants of waves of different periods. This study is recorded in the remaining portion of this section. This section is also entirely new.

The work given in § I and a portion of that given in § II, together with a description of experimental investigations bearing upon the same, were published in vol. XVI of the Transactions of the American Institute of Electrical Engineers for 1899. Additional experimental investigations bearing upon the principal problem, that is the problem of § III, will be published in the near future.

§ I. Wave Propagation along a uniform linear Conductor.

A. *Waves of forced period.*

The conductor is a loop of wire AB (Fig. 1). At one point of the loop is a transmitting apparatus A, at the opposite point is a receiving apparatus B. The distance between A and B is l, equal to one-half the length of the whole loop. The distance of any element ds from A is s.

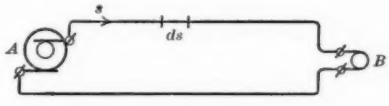

Fig. 1

Let L, R, and C, denote the inductance, resistance, and capacity, respectively, per unit length. Let y be the current at any point s. Then

$$L\frac{d^2y}{dt^2} + R\frac{dy}{dt} = \frac{1}{C}\frac{\partial^2 y}{\partial s^2} \tag{1}$$

is the well known equation of propagation.

To formulate the two boundary conditions, one at the receiving and the other at the transmitting apparatus, let L_0, R_0, C_0, and L_1, R_1, C_1, respectively be the inductance, resistance, capacity, of the transmitting and of the receiving apparatus. Each contains a condenser in series with its inductance and resistance. Let the e. m. f. impressed by A be of type Ee^{ipt}. The boundary equations can now be expressed as follows:

$$\begin{cases} -2\left(\dfrac{\partial y}{\partial s}\right)_{s=0} = (D_0 - h_0 y)_{s=0} & \text{at transmitting apparatus,} \\ +2\left(\dfrac{\partial y}{\partial s}\right)_{s=l} = -(h_1 y)_{s=l} & \text{at receiving apparatus,} \end{cases} \tag{2}$$

where

$$D_0 = ipC_0Ee^{ipt},$$

$$h_0 = C_0(-p^2\lambda_0 + ipR_0), \qquad h_1 = C_1(-p^2\lambda_1 + ipR_1),$$

$$\lambda_0 = L_0 - \frac{1}{p^2C_0}, \qquad \lambda_1 = L_1 - \frac{1}{p^2C_1}.$$

The physical character of the problem suggests the following solution:

$$y = K_1\cos\mu\xi + K_2\sin\mu\xi, \tag{3}$$

where $\xi = l - s$, and K_1 and K_2 are proportional to e^{ipt}.

Equation (1) is satisfied for all values of K_1 and K_2 if

$$C(-p^2L + ipR) = -\mu^2 = -(a + i\beta)^2.$$

From this we get

$$a = \sqrt{\tfrac{1}{2}pC\{\sqrt{p^2L^2 + R^2} + pL\}},$$

$$\beta = \sqrt{\tfrac{1}{2}pC\{\sqrt{p^2L^2 + R^2} - pL\}}. \tag{4}$$

When pL is large in comparison with R,

$$a = p\sqrt{LC}, \qquad \beta = \frac{R}{2}\sqrt{\frac{C}{L}}.$$ *

* Observe that in the case of a twin conductor where the mutual capacity is large in comparison with the capacity of each conductor to earth we must use $2R$ and $2L$ in place of R and L.

The boundary equations will be satisfied if

$$K_1 = \frac{2\mu D_0}{F}, \qquad K_2 = \frac{h_1 D_0}{F},$$

where

$$F = (h_0 h_1 - 4\mu^2) \sin \mu l + 2\mu(h_0 + h_1) \cos \mu l .$$

Hence (3) can now be written:

$$y = (2\mu \cos \mu\xi + h_1 \sin \mu\xi)\frac{D_0}{F}. \tag{5}$$

When $h_0 = h_1 = 0$,

$$y = -\frac{D_0 \cos \mu\xi}{2\mu \sin \mu l}. \tag{5a}$$

Equation (5) is a complete solution for the propagation of waves of forced period. It represents simple harmonic damped waves. The most essential elements which enter into the description of such waves are the *wave-length* λ and the *damping* or *attenuation factor*. These can easily be calculated. Since $\mu = a + i\beta$, we shall have $\lambda = 2\pi/a$; the attenuation factor is $e^{-\beta\xi}$.

It is evident that β diminishes as L increases. A high reactance per unit length means small attenuation and a slow speed of propagation.

B. *Waves with a natural period.*

Free oscillations are readily calculated for a few special cases. Equation (5) is a general solution for free oscillations also, provided, however, that μ has such a value as to make $F = 0$, since $D_0 = 0$; that is, we must have

$$(h_0 h_1 - 4\mu^2) \sin \mu l + 2\mu(h_0 + h_1) \cos \mu l = 0 ; \tag{6}$$

but, of course, in this case,

$$h_0 = kC(k\lambda_0 + R_0), \qquad h_1 = kC(k\lambda_1 + R_1), \qquad -\mu^2 = kC(kL + R).$$

Equation (6) is a transcendental equation and can be readily solved in a few simple cases.

Case 1.—The transmitting and receiving apparatus are not present.

In this case $h_0 = h_1 = 0$. Equation (6) reduces to

$$\sin \mu l = 0 .$$

Therefore

$$\mu = s\pi/l ,$$

where s can have any integral value from 1 to ∞. The periods of free oscillation are calculated from the equation:

$$-\mu^2 = k^2LC + kRC = -\frac{s^2\pi^2}{l^2}.$$

Therefore

$$k = -\frac{R}{2L} \pm \sqrt{-1}\sqrt{\frac{1}{LC}\frac{s^2\pi^2}{l^2} - \frac{R^2}{4L^2}} = -\frac{R}{2L} \pm ik_s.$$

There is, therefore, an infinite number of periods which are harmonically related to each other unless the damping constant $R/2L$ is not negligibly small in comparison with π^2/l^2LC.

The most general solution of this case can be written:

$$y = e^{-\frac{Rt}{2L}} \sum_{s=1}^{\infty} A_s \cos\left(\frac{s\pi}{l}\xi\right) \cos\left(k_s t - \epsilon_s\right). \tag{7}$$

The wave-lengths are $2l/1$, $2l/2$, $2l/3$, $\cdots$, $2l/s$, $\cdots$.

Case 2.—The transmitting apparatus is not present and in place of the receiving apparatus there is a break in the wire.

In this case $h_0 = 0$, $h_1 = \infty$. Equation (6) reduces to

$$\cos \mu l = 0.$$

Therefore

$$\mu = \frac{2s+1}{l}\frac{\pi}{2},$$

$$k = -\frac{R}{2L} \pm i\sqrt{\frac{1}{LC}\left(\frac{2s+1}{l}\frac{\pi}{2}\right)^2 - \frac{R^2}{4L^2}} = -\frac{R}{2L} + ik_{2s+1},$$

$$y = e^{-\frac{R}{2L}t} \sum_{s=0}^{\infty} A_{2s+1} \sin\left(\frac{2s+1}{l}\frac{\pi}{2}\xi\right) \cos\left(k_{2s+1} - \epsilon_{2s+1}\right). \tag{8}$$

The wave-lengths are $4l/1$, $4l/3$, $4l/5$, $\cdots$.

The damping factor is in both cases the same for all frequencies; hence the color of the complex harmonic vibration remains unchanged during the whole epoch while the vibrations last.

§ II. Electrical Oscillations on a non-uniform Conductor of the first Type.

The conductor consists of $2n$ equal coils, L_1, L_2, $\cdots$, L_n (Fig. 2) connected in series so as to form a closed loop. At one point A of this loop is an alternator, at the opposite point is a receiving apparatus B. At equal distances $2(n-1)$ equal condensers, C_1, $\cdots$, C_{n-1}, connect the conductor to ground. The whole loop

is thus divided into $2n-2$ component circuits, $1, 2, \cdots, 2n-2$. It is evident that in the limit when n becomes infinitely large, this conductor becomes an ordinary telegraph or telephone line with uniformly distributed resistance, capacity, and inductance. The question now arises: *Under what conditions will a conductor of this kind become equivalent, with sufficient approximation, to a uniform line, even when n is not infinitely large?* This problem does not seem to have been solved before. In its main features it is similar to that which LAGRANGE solved in his *Mécanique Analytique*, pt. II, sec. VI, the problem, namely, of the free vibrations of a weightless string, fixed at its two

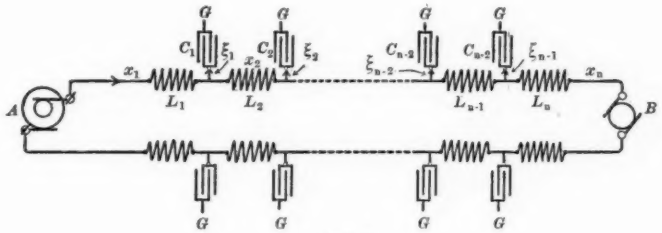

FIG. 2

ends, and loaded at equidistant points by equal weights. But it is much more general because frictional resistances are taken into consideration, and also because forced as well as free oscillations are considered.

Let L_0 and L_1, R_0 and R_1, C_0 and C_1 be the inductance, ohmic resistance, capacity of A and B, respectively. Let L, R, C be the corresponding quantities of the coils and condensers in the several component circuits.

The real part of Ee^{ipt} is the e. m. f. impressed by alternator A.

Let $x_1, \cdots, x_n$ be the currents in the component circuits; $P_1, \cdots, P_{n-1}$ the differences of potential in the line condensers, $C_1, C_2, \cdots, C_{n-1}$; P_0 and P' the differences of potential in the condensers in A and B respectively; and $\xi_1, \cdots, \xi_{n-1}$ the condenser currents.

We shall have

$$(1) \qquad \begin{cases} \xi_1 = C\dfrac{dP_1}{dt}, \quad \xi_2 = C\dfrac{dP_2}{dt}, \quad \cdots, \\ \xi_1 = x_1 - x_2, \quad \xi_2 = x_2 - x_3, \quad \cdots. \end{cases}$$

A. *Forced oscillations.*

Stating the law of equality of action and reaction for each component circuit, we obtain the following n differential equations:

$$(2)\qquad \begin{cases} (L_0+2L)\dfrac{dx_1}{dt}+(R_0+2R)x_1+2P_1+P_0=Ee^{ipt}, \\ L\dfrac{dx_2}{dt}+Rx_2+P_2-P_1=0, \\ \cdots\cdots\cdots\cdots\cdots \\ L\dfrac{dx_{n-1}}{dt}+Rx_{n-1}+P_{n-1}-P_{n-2}=0, \\ (L_1+2L)\dfrac{dx_n}{dt}+(R_1+2R)x_n-2P_{n-1}+P'=0. \end{cases}$$

When the steady state has been reached, the currents will be, just like the impressed e. m. f., simple harmonics of the time t, that is,

$$(3)\qquad x_1=A_1e^{ipt},\quad x_2=A_2e^{ipt},\ \cdots,$$

where A_1, A_2, $\cdots$ are complex quantities.

Differentiating each member of (2) and substituting from (3) and (1) we obtain:

$$(4)\qquad \begin{cases} hx_1+\xi_1-0=D, \\ hx_2+\xi_2-\xi_1=0, \\ \cdots\cdots\cdots \\ hx_{n-1}+\xi_{n-1}-\xi_{n-2}=0, \\ hx_n+0-\xi_{n-1}=-h_1x_n, \end{cases}$$

where

$$h=C(-p^2L+ipR),\qquad h_1=\tfrac{1}{2}C(-p^2\lambda_1+ipR_1).$$

$$(5)\qquad D=\tfrac{1}{2}ipCEe^{ipt}-\tfrac{1}{2}C(-p^2\lambda_0+ipR_0)=D_0-h_0x_1,$$

$$\lambda_0=L_0-\frac{1}{p^2C_0},\quad \lambda_1=L_1-\frac{1}{p^2C_1}.$$

Another form is obtained by substituting for ξ_1, ξ_2, $\cdots$ as follows:

$$(6)\qquad \begin{cases} (h+1)x_1+0-x_2=D=D_0-h_0x_1, \\ (h+2)x_2-x_1-x_3=0, \\ \cdots\cdots\cdots \\ (h+2)x_{n-1}-x_{n-2}-x_n=0, \\ (h+1)x_n-x_{n-1}-0=-h_1x_n. \end{cases}$$

Equation (3) of § I suggests the following solution:

$$x_m = K_1 \cos 2(n-m)\theta + K_2 \sin 2(n-m)\theta . \tag{6a}$$

If $h + 2 = 2\cos 2\theta$, then all the equations except the first and the last will be satisfied for all values of K_1 and K_2. These two equations, which correspond to the boundary equations of § I, will be also satisfied if we assign to K_1 and K_2 suitable values, as follows:

$$K_2 = \frac{(h_1 - 1) + \cos 2\theta}{\sin 2\theta} K_1 ,$$

$$K_1 = \frac{D_0 \sin 2\theta}{h_0 h_1 \sin 2(n-1)\theta - 4\sin^2\theta \sin 2n\theta + 2(h_0 + h_1)\sin\theta \cos(2n-1)\theta} .$$

We can now write

$$x_m = \frac{[2\sin\theta \cos(2n - 2m + 1)\theta + h_1 \sin 2(n-m)\theta] D_0}{h_0 h_1 \sin 2(n-1)\theta - 4\sin^2\theta \sin 2n\theta + 2(h_0 + h_1)\sin\theta \cos(2n-1)\theta} . \tag{7}$$

As θ is a complex angle, the forced oscillations of type e^{ipt} on a non-uniform conductor of this kind are simple harmonic damped oscillations. The similarity between this conductor and a uniform wire will be discussed presently.

When $h_0 = h_1 = 0$ we obtain

$$x_m = -\frac{D_0 \cos(2n - 2m + 1)\theta}{2\sin\theta \sin 2n\theta} . \tag{7a}$$

B. *Free oscillations.*

Equation (7) holds for free as well as for forced oscillations. But since $D_0 = 0$ in the case of free oscillations, it follows that the denominator of (7) must vanish to prevent the vanishing of all currents. We shall therefore have

$$h_0 h_1 \sin(2n-2)\theta - 4\sin^2\theta \sin 2n\theta + 2(h_0 + h_1)\sin\theta \cos(2n-1)\theta = 0 . \tag{8}$$

From this equation, θ and the corresponding periods and damping constants have to be determined. A solution can be readily obtained for a small number of cases; the two most important will be considered here.

First case. The transmitting and the receiving apparatus are not present. In this case $h_0 = h_1 = 0$, and

$$x_m = B\cos(2n - 2m + 1)\theta . \tag{9}$$

It is found from (8) that (9) is actually the solution of the differential equations (6) for $h_0 = h_1 = D_0 = 0$, provided that

$$\theta = \frac{s\pi}{2n},$$

where s is any integer from 1 to $2n$.

Hence the most general solution will be:

$$x_m = \sum_{s=1}^{2n} B_s \cos(2n - 2m + 1)\frac{s\pi}{2n}. \tag{10}$$

But it should be observed now that x_m is a periodic function of the time, that is,

$$x_m = \sum K_s e^{p_s t}.$$

The constant p_s, which measures the period and the damping constant of the free oscillation, is determined from the relation:

$$h = -4\sin^2\theta.$$

In the case of free oscillations,

$$h = p^2LC + pRC, \quad \theta = \frac{s\pi}{2n},$$

Hence

$$p_s^2LC + p_sRC = -4\sin^2\frac{s\pi}{2n}. \tag{11}$$

Before solving this equation, it is desirable to make the following substitutions:—

Let L', C', and R' be the total inductance, capacity, and resistance, respectively, of one-half of the conductor, then

$$L = \frac{L'}{n}, \qquad C = \frac{C'}{n}, \qquad R = \frac{R'}{n}.$$

Let l denote the half length of a uniform wire having σ, r, and c, for inductance, resistance, and capacity per unit length, and let it have the same total inductance, resistance, and capacity as the non-uniform conductor. Since

$$l\sigma = L', \quad lr = R', \quad lc = C',$$

we shall have

$$L = \frac{l\sigma}{n}, \quad C = \frac{lc}{n}, \quad R = \frac{lr}{n}.$$

This uniform wire having the same total inductance, resistance, and capacity as the non-uniform conductor, will be called the *corresponding uniform conductor.*

From (11) we obtain

$$\frac{l^2}{n^2}(p_s^2\sigma c + p_s cr) = -4\sin^2\frac{s\pi}{2n},$$

whence

$$p_s = -\frac{r}{2\lambda} \pm \sqrt{-1}\sqrt{\frac{1}{\sigma c}\frac{4n^2}{l^2}\sin^2\frac{s\pi}{2n} - \frac{r^2}{4\lambda^2}} = -\frac{r}{2\lambda} \pm ik_s .$$

Equation (10) now becomes:

$$x_m = e^{-\frac{rt}{2\sigma}} \sum_{s=1}^{2n} A_s \cos(2n - 2m + 1)\frac{s\pi}{2n} \cos(k_s t - \epsilon_s) . \tag{12}$$

It is clear that the oscillations on the non-uniform conductor of the first type have the same damping constant as the oscillations on the corresponding uniform conductor. It will be shown presently that under certain conditions they will also have the longer periods, up to a certain limit, very nearly the same as the corresponding uniform conductor. For these periods, then, the non-uniform conductor will be equivalent to its corresponding uniform conductor.

Second case. The transmitting apparatus is not present, and in place of the receiving apparatus there is a break in the line at B.

In this case $h_0 = 0$, $h_1 = \infty$. Equation (9) gives:

$$x_m = B \sin(2n - m + 2)\theta ,$$

provided that

$$\cos(2n - 1)\theta = 0 ,$$

or

$$\theta = \frac{2s + 1}{2n - 1}\frac{\pi}{2} .$$

We shall have, therefore,

$$p_{2s+1} = -\frac{r}{2\sigma} \pm i\sqrt{\frac{1}{\lambda c}\frac{4n^2}{l^2}\sin^2\frac{2s + 1}{2n - 1}\frac{\pi}{2} - \frac{r^2}{4\lambda^2}} = -\frac{r}{2\sigma} \pm ik_{2s+1} .$$

Hence

$$x_m = e^{-\frac{rt}{2\sigma}} \sum_{s=0}^{2n} A_{2s+1} \sin(2n - 2m + 2)\frac{2s + 1}{2n - 1}\frac{\pi}{2} \cos(k_{2s+1}t - \epsilon_{2s+1}) . \tag{13}$$

The remark in the preceding case regarding the damping constant and the free periods also applies here.

The wave-lengths of free oscillations.—The angles $s\pi/2n$ and $(2s + 1)\pi/2(2n - 1)$ have an interesting physical meaning which will be brought out by considering the wave-lengths of the oscillations. Consider one of the component harmonics of x_m in Case I, say

$$\xi_{m,s} = A_s \cos(2n - 2m + 1)\frac{s\pi}{2n} \cos(k_s t - \epsilon_s) .$$

Compare it with the corresponding component of x_{m_1}, that is, with

$$\xi_{m_1,s} = A_s \cos(2n - 2m_1 + 1)\frac{s\pi}{2n} \cos(k_s t - \epsilon_s) .$$

If they are a wave-length apart, $\xi_{m,s} = \xi_{m_1,s}$, and $m_1 - m$ is equal to the number of coils covered by one wave-length. But in this case,

$$(2n - 2m + 1)\frac{s\pi}{2n} = (2n - 2m_1)\frac{s\pi}{2n} + 2\pi;$$

therefore

$$m_1 - m = \frac{2n}{s} = \nu_s.$$

Hence ν_s is the number of coils covered by a wave-length which corresponds to the harmonic s. It can be shown that, in the second case, $2(2n-1)/2s+1 = \nu_s$. For $s\pi/2n$ and $(2s+1)\pi/2(2n-1)$ we can, therefore, write $\pi/\nu_s = 2\pi/2\nu_s$. The physical meaning of $2\pi/\nu_s$ can now be readily fixed. A coil represents a definite fraction of a wave-length, and this fraction will have a different value for different harmonics. The higher the harmonic the shorter will be the wave-length and therefore, the larger will be the value of this fraction. It is convenient, however, to measure this fraction in terms of an angle instead of in terms of a wave-length. If we arbitrarily assume that an angular distance 2π corresponds to a wave-length then an angular distance $2\pi/n$ will correspond to the nth part of a wave-length. With this understanding,

$$s\pi/2n \text{ and } (2s+1)\pi/2(2n-1),$$

that is $2\pi/2\nu_s$ represents one half of the angular distance covered by a coil.

C. *A non-uniform conductor of the first type compared with its corresponding uniform wire.*

a. Similarity with respect to free oscillations.—Comparing the expressions for the free periods of oscillation which were obtained in sections I and II we see that as long as π/ν_s can be written for $\sin \pi/\nu_s$, so long will the periods of free oscillations of the non-uniform conductor be nearly the same as those of its corresponding uniform conductor. We have therefore the simple rule: *A non-uniform conductor of the first type represents its corresponding uniform conductor as nearly as one half of the angular distance covered by one of its coils represents the sine of that distance.* The non-uniform conductor employed by me* in my experiments had 400 coils. In this case

$$\frac{\pi}{\nu_s} = \frac{s\pi}{400}.$$

For $s = 25$, we have $\pi/\nu_s = \pi/16$. Now $\sin \pi/16$ differs from $\pi/16$ by $\frac{2}{3}$ of one per cent of the value of $\pi/16$. Hence the period of the 25th harmonic of my non-

* See paper cited in the introduction.

uniform conductor differs from that of the same harmonic on the corresponding uniform wire by less than $\frac{2}{3}$ of one per cent. For lower harmonics the difference is smaller. The 25th harmonic had approximately 3,500 p. p. s. Hence up to the 25th harmonic, that is up to 3,500 p. p. s., the non-uniform conductor employed by me had nearly the same free periods as its corresponding uniform conductor.

b. Similarity with respect to forced oscillations.—The wave-length and the attenuation constant corresponding to a given frequency speed p_s can be studied by studying the angle θ from the following equation:

$$-4\sin^2\theta_s = -p_s^2 LC + ip_s RC = \frac{l^2}{n^2}\left[-p_s^2\rho C + ip_s rC\right]$$
$$= -\frac{l^2}{n^2}\mu_s^2 = -\frac{l^2}{n^2}(a_s + i\beta_s)^2,$$

or

$$\sin\theta_s = \tfrac{1}{2}\frac{l}{n}(a_s + i\beta_s),$$

where ρ, C, and r, are the inductance, capacity, and resistance, respectively, per unit length of the corresponding uniform conductor; and μ_s, a_s, and β_s have the same meanings as in § I.

Referring now to equation (4), § I, we shall find that

$$a_s \geqq \beta_s.$$

If, therefore, $la_s/2n$ is sufficiently small we can put

$$\theta_s = \tfrac{1}{2}\frac{l}{n}(a_s + i\beta_s) = \tfrac{1}{2}\frac{l}{n}\mu_s.$$

When this substitution is made in (7) and (7a) these equations will transform into (5) and (5a) of § I, which shows that under these conditions the non-uniform conductor of the first type becomes approximately equivalent to its corresponding uniform conductor, the degree of approximation being the same as that of $\sin\theta_s$ to θ_s.

The physical meaning of this can readily be made clear. If by λ_s we denote the wave-length on the corresponding uniform conductor, corresponding to the frequency speed p_s, then according to § I,

$$a_s = \frac{2\pi}{\lambda_s}, \quad \frac{l}{n}a_s = \frac{l}{n}\frac{2\pi}{\lambda_s}.$$

Let n_s be the number of coils on the non-uniform conductor which cover a wave-length of frequency speed p_s, and let ϕ_s be the angular distance covered by a coil, then

$$\phi_s = \frac{2\pi}{n_s}.$$

Again

$$n : n_s :: l : \lambda_s ,$$

or

$$n : \frac{2\pi}{\phi_s} :: l : \lambda_s .$$

Hence

$$\frac{l}{n}\frac{2\pi}{\lambda_s} = \frac{l}{n}\alpha_s = \phi_s .$$

The substitution mentioned above will therefore be permissible when $\frac{1}{2}\phi_s$ (that is, one-half of the angular distance covered by a coil) is approximately equal to sin $\phi_s/2$. We have here the same rule as in the case of free oscillations. It will be shown in § III that this rule is also applicable to non-uniform conductors of the second type, for waves of both forced and free periods.

§ III. Wave Propagation along a non-uniform Conductor of the second Type.

A. *Waves of forced periods.*

The conductor is represented in the diagram of Fig. 3. A long uniform wire forms a loop $L_1 \cdots L_{k+2}$. At equal intervals are inserted in series $2k + 2$ equal

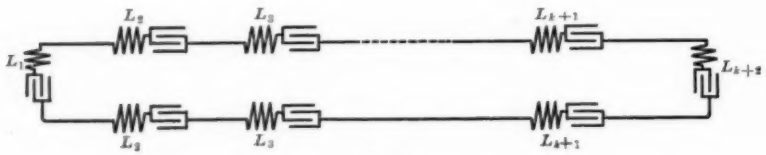

Fig. 3

coils $L_1, L_2, \cdots$. To make the discussion more general, let each coil have a condenser of capacity C_0 in series with it.

Let L, R, and C, represent the inductance, resistance and capacity, respectively, per unit length of the uniform wire. Let an e. m. f. of type Ee^{ipt} be impressed at L_1. Employing the usual notation we shall have the following equation of propagation at every point of the uniform wire:

$$L\frac{d^2y}{dt^2} + R\frac{dy}{dt} = \frac{1}{C}\frac{\partial^2 y}{\partial s^2}, \tag{1}$$

the same as in § I, but here the integral has to satisfy $2k + 2$ boundary conditions, one at each coil. On account of the symmetry of distribution of the coils on the two sides of the loop this number is reduced to $k + 2$.

To formulate these boundary conditions introduce the following notation:

Let $2l$ be the length of the loop; $l/(k+1)$ will be the distance between two consecutive coils. Let L_0, R_0, and C_0, be the inductance, resistance, and capacity, respectively, of each coil. Let y_m be the current at any point of the interval between the coils L_m and L_{m+1}. Denote the distance of this point from coil L_m by ξ.

The currents in the coils will be denoted by $x_1, x_2, \cdots, x_{k+1}, x_{k+2}$. It is evident that

$$(y_m)_{\xi=0} = (y_{m-1})_{\xi=l/(k+1)} = x_m.$$

The boundary conditions for all coils except L_1 and L_{k+2} can now be stated. They are:

$$\left(\frac{\partial y_{m-1}}{\partial \xi}\right)_{\xi=l/(k+1)} - \left(\frac{\partial y_m}{\partial \xi}\right)_{\xi=0} = -C(-p^2\lambda_0 + ipR_0)x_m = -hx_m,$$

where

$$\lambda_0 = L_0 - \frac{1}{p^2 C_0},$$

There are k equations of this form. The boundary equations for the first and the last coil are of the following forms:

$$-2\left(\frac{\partial y_1}{\partial \xi}\right)_{\xi=0} = ipCEe^{ipt} - hx_1 = D - hx_1$$

for the first coil, and

$$2\left(\frac{\partial y_{k+1}}{\partial \xi}\right)_{\xi=l/(k+1)} = -hx_{k+2}$$

for the last coil.

Equation (3) of § I and equation (6a) of § II suggest the following trial solution:

(1a) $$y_m = K_1 \cos \mu\xi + K_2 \sin \mu\xi,$$

where μ has the same value as in § I.

When $\xi = 0$, y_m becomes x_m, and when $\xi = l/(k+1)$, y_m becomes x_{m+1}. Hence

$$x_m = K_1, \quad x_{m+1} = K_1 \cos\frac{\mu l}{k+1} + K_2 \sin\frac{\mu l}{k+1}.$$

Therefore

(2) $$y_m = \frac{x_m \sin\mu\left(\frac{l}{k+1} - \xi\right) + x_{m+1}\sin\mu\xi}{\sin\frac{\mu l}{k+1}}.$$

We have now to determine the $k+2$ constants $x_1, x_2, \cdots, x_{k+2}$ from the $k+2$ boundary equations.

Begin with the boundary equation at the coil L_1:

$$\frac{\partial y_m}{\partial \xi} = \frac{\mu\left\{ - x_m \cos \mu\left(\frac{l}{k+1} - \xi\right) + x_{m+1} \cos \mu\xi \right\}}{\sin \frac{\mu l}{k+1}}.$$

Therefore

$$-2\left(\frac{dy_1}{d\xi}\right)_{\xi=0} = 2\left(x_1 \cos \frac{\mu l}{k+1} - x_2\right) \frac{\mu}{\sin \frac{\mu l}{k+1}} = D - hx_1.$$

Let

$$\sigma = h \frac{\sin \frac{\mu l}{k+1}}{\mu} - 4 \sin^2 \tfrac{1}{2} \frac{\mu l}{k+1}, \qquad \rho = \frac{\sin \frac{\mu l}{k+1}}{\mu} D;$$

the boundary equations can then be written:

$$(3) \quad \begin{cases} (\sigma+1)x_1 - 0 - x_2 = \dfrac{\rho}{2} + \dfrac{\sigma}{2} x_1, \\ (\sigma+2)x_2 - x_1 - x_3 = 0, \\ (\sigma+2)x_3 - x_2 - x_4 = 0, \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ (\sigma+2)x_m - x_{m-1} - x_{m+1} = 0, \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ (\sigma+2)x_{k+1} - x_k - x_{k+2} = 0, \\ (\sigma+1)x_{k+2} - x_{k+1} - 0 = \dfrac{\sigma}{2} x_{k+2}. \end{cases}$$

This system is of the same form as system (6) of §II. Equation (7) of that section enables us to write down the solution of (3).

Let

$$\sigma + 2 = 2 \cos 2\psi;$$

then x_m, y_m have the respective values:

$$(4) \quad - \frac{D \sin \frac{\mu l}{k+1} \cos 2\,(k-m+2)\psi}{2\mu \sin 2\psi \sin 2\,(k+1)\psi},$$

$$(5) \quad - \frac{\left[\cos 2(k-m+2)\psi \sin \mu\left(\frac{l}{k+1} - \xi\right) + \cos 2(k-m+1)\psi \sin \mu\xi\right] D}{2\mu \sin 2\psi \sin 2(k+1)\psi}.$$

These statements give the complete mathematical representation of waves of forced period on a conductor of this kind. The angles ψ and μ are, of course, complex; hence, in order to calculate from these equations in any particular case the principal quantities with which we are concerned in wave propagation, namely, the wave-length and the attenuation constant, it would be necessary to study separately the real and the imaginary parts of these equations. This study in its broadest aspect presents considerable mathematical complexity. It is, however, somewhat remote from the principal aim of this paper which is to ascertain the conditions under which a non-uniform conductor of the second type is approximately equivalent to its corresponding uniform conductor, that is, to ascertain the frequencies for which the two will have approximately the same wave-lengths and the same attenuation constants.

Equivalence of a non-uniform conductor of the second type to its corresponding uniform conductor.

The solution of this problem is obtained by studying the following equation:

$$\frac{h_s \sin \dfrac{\mu_s l}{k+1}}{\mu_s} + 2 \cos \frac{\mu_s l}{k+1} = 2 \cos 2\psi_s,$$

or

$$\frac{h_s \sin \dfrac{\mu_s l}{k+1}}{\mu_s} - 4 \sin^2 \tfrac{1}{2} \frac{\mu_s l}{k+1} = -4 \sin^2 \psi_s.$$

The subscript s denotes that the particular frequency speed p_s is considered, and

$$\mu_s = a_s + i\beta_s, \quad a_s \geqq \beta_s.$$

If λ_s is the wave-length corresponding to p_s on the uniform wire before the introduction of the reactance points, then

$$\frac{l}{k+1} a_s = \frac{l}{k+1} \frac{2\pi}{\lambda_s}.$$

Let θ_s be the angular distance between two consecutive coils for wave-length λ_s: then

$$2\pi : \lambda_s :: \theta_s : \frac{l}{k+1};$$

therefore

$$\theta_s = \frac{l}{k+1} \frac{2\pi}{\lambda_s} = \frac{l}{k+1} a_s.$$

Hence if θ_s is so small that $\sin \theta_s = \theta_s$ very nearly, then

$$\frac{l}{k+1} h_s - \left(\frac{l}{k+1}\right)^2 \mu_s^2 = -4 \sin^2 \psi_s ,$$

or

$$\left(\frac{l}{k+1}\right)^2 C \{-p_s^2(\rho + L) + ip(r + R)\} = -4 \sin^2 \psi_s ,$$

where ρ and r are the inductance and resistance, respectively, per unit length of the inserted coils, so that $\rho + L$ and $r + R$, are the inductance and resistance respectively, of the corresponding uniform conductor. This last equation can also be written:

$$\tfrac{1}{2} \frac{l}{k+1} (a_s' + i\beta_s') = \sin \psi_s .$$

In this case also $a_s' \cong \beta_s'$.

Let λ_s' be the wave-length for frequency speed p_s on the corresponding uniform conductor. Then

$$a_s' = \frac{2\pi}{\lambda_s'} .$$

Put $\phi_s' = la_s'/(k+1)$; then ϕ_s' is the angular distance between two consecutive coils, the angular distance of λ_s' being 2π.

If, therefore, $\frac{1}{2}\phi_s'$ is so small that $\sin \frac{1}{2}\phi' = \frac{1}{2}\phi'$ very nearly, then

$$\psi_s = \tfrac{1}{2} \frac{l}{k+1} (a_s' + i\beta_s') = \tfrac{1}{2} \frac{l}{k+1} \mu_s' ,$$

very nearly.

Making this substitution in (4) we obtain equation (5a) of § I. The equation of wave propagation for a non-uniform conductor of the second type is, under these conditions, the same as that for its corresponding uniform conductor. The two will have approximately the same wave-length and the same attenuation constant for the frequency speed p_s.

The degree of approximation is the same as that of $\sin \psi_s/2$ to $\psi_s/2$.

A numerical example will illustrate this point more clearly.

Consider a uniform telegraph wire having the following constants per mile:

$$L = .004 \text{ Henry},$$
$$R = 7 \text{ Ohm},$$
$$C = .01 \times 10^{-6} \text{ Farads}.$$

Introduce at each mile a coil having

$$L_1 = .036 \text{ Henry},$$
$$R = 1 \text{ Ohm}.$$

The non-uniform conductor thus obtained will have for its corresponding uniform conductor a uniform wire having, per mile,

$$L + L_1 = .04 \text{ Henry},$$
$$R + R_1 = 8 \text{ Ohm},$$
$$C = .01 \times 10^{-6} \text{ Farads}.$$

For a frequency of 1500 p. p. s. we shall have $p_s = 2\pi \times 1500$.

The wave-length λ'_s on the corresponding uniform conductor for this frequency is very nearly

$$\lambda'_s = 33 \text{ miles}.$$

Hence, since the interposed coils are one mile apart, we shall have

$$\sin \tfrac{1}{2}\phi' = \sin \frac{\pi}{33}.$$

Now $\sin \pi/33 = \pi/33$ to within $\frac{1}{6}$ of one per cent of the value of $\pi/33$. Up to this degree of approximation the non-uniform conductor just described will have the same wave-length and the same attenuation constant as its corresponding uniform conductor. For lower frequencies the degree of approximation will be much higher; hence for all frequencies which are of any importance in the telephonic transmission of speech the two conductors are equivalent.

B. *Waves of free period.*

Equation (4) holds for all values of D; it should therefore hold good when $D = 0$. In this case, however, all the currents will vanish unless the denominator vanishes, that is, unless

$$\sin 2\psi \sin 2(k+1)\psi = 0. \tag{4a}$$

This transcendental equation together with the equation:

$$\frac{h}{\mu} \sin \frac{\mu l}{k+1} + 2 \cos \frac{\mu l}{k+1} = 2 \cos 2\psi, \tag{4b}$$

determines the free periods and their corresponding damping constants.

Equation (4a) will be satisfied when

$$\psi = \frac{r\pi}{2(k+1)},$$

where r is any integer from 0 to ∞.

When $r = s(k+1)$ we shall have not only $\sin 2(k+1)\psi = 0$, but also $\sin 2\psi = 0$.

These particular values of ψ give a set of free oscillations which form a distinct group, called further below the oscillations of *normal period*.

Equation (4*b*) should therefore be written:

$$\frac{h_r \sin \frac{\mu_r l}{k+1}}{\mu_r} + 2\cos\frac{\mu_r l}{k+1} = 2\cos\frac{r\pi}{k+1}. \tag{4c}$$

This equation will be referred to as the *period equation*.

It should be observed that

$$-\mu_r^2 = C(\nu_r^2 L + \nu_r R), \quad h_r = C(\nu_r^2 \lambda_0 + \nu_r R_0).$$

An inspection of (5) shows that the complete solution for waves of free period can now be written:

$$y_m = \sum_{r=1}^{\infty} \frac{A_r}{\sin\frac{r\pi}{k+1}} \left\{ \cos(k-m+2)\frac{r\pi}{k+1} \sin\mu_r\left(\frac{l}{k+1}-\xi\right) + \cos(k-m+1)\frac{r\pi}{k+1}\sin\mu_r\xi \right\},$$

or

$$y_k = \sum_{r=1}^{\infty} \eta_r.$$

In the case of a uniform wire the periods and the wave-lengths of the simple harmonic components $\eta_1, \eta_2, \cdots, \eta_r, \cdots$ have in general a harmonic relation. The introduction of the reactance points $L_1, L_2, \cdots$ disturbs this relation in consequence of a displacement in the values of the normal periods and of the normal damping constant. It is evident from purely physical considerations that since the introduction of the reactance points increases, in general, the inductance per unit length, the periods will be lengthened.

a. Waves of normal period.

According to equation (7), § I the wave-lengths of free oscillations on a uniform loop are as follows:

$$\frac{2l}{1}, \frac{2l}{2}, \frac{2l}{3}, \cdots, \frac{2l}{k+1}, \frac{2l}{k+2}, \cdots, \frac{2l}{2(k+1)}, \frac{2l}{2(k+3)},$$
$$\cdots, \frac{2l}{s(k+1)}, \frac{2l}{s(k+1)+1}, \cdots.$$

Divide this series into groups of $(k+1)$ members each. Consider now the last member of each group; call it the *terminal member*. The wave-lengths of the terminal members are

$$\frac{2l}{(k+1)}, \frac{2l}{2(k+1)}, \cdots, \frac{2l}{s(k+1)}, \cdots.$$

These terminal members will appear with normal periods among the free oscillations on the periodically loaded loop. To show this consider the components of y_m of type $\eta_{s(k+1)}$ where s may be any integer between 1 and ∞. Since in the case of these components $\psi = s\pi$, the period equation becomes:

$$\frac{h}{\mu} \sin \frac{\mu l}{k+1} + 2 \cos \frac{\mu l}{k+1} = 2 \cos s\pi.$$

This equation will be satisfied by putting

$$\mu = \frac{(k+1)s\pi}{l}.$$

The free periods of components of type $\eta_{s(k+1)}$ are now easily obtained:

$$\nu_s^2 LC + \nu_s RC = -\mu^2 = -\frac{(k+1)^2 s^2 \pi^2}{l^2}.$$

Therefore

$$\nu_s = -\frac{R}{2L} \pm i \sqrt{\frac{1}{LC} \frac{(k+1)^2 s^2 \pi^2}{l^2} - \frac{R^2}{4L^2}} = -\frac{R}{2L} \pm i p_s.$$

From the expression for y_m we deduce:

$$\eta_{s(k+1)} = B_{s(k+1)} \sin \frac{(k+1)s\pi\xi}{l},$$

where

$$B_{s(k+1)} = B'_{s(k+1)} e^{-\frac{R}{2L}t} \cos(p_s t - \epsilon_s).$$

The wave-length of this component is $2l/s(k+1)$.

The wave-lengths of the component of type $\eta_{s(k+1)}$ are

$$\frac{2l}{k+1}, \frac{2l}{2(k+1)}, \frac{2l}{3(k+1)}, \cdots, \frac{2l}{s(k+1)}, \cdots.$$

From the wave-lengths and their corresponding periods we see that the oscillations of the type $\eta_{s(k+1)}$ are identical with the terminal members of the groups into which the free oscillations on a uniform loop were divided above. The physical reason why these periods have not been disturbed by the introduction of reactance points is easily seen; it is because they have their nodes at the reactance points. The other k members of each group have been disturbed by the presence of the reactance points. The character of this disturbance will now be discussed.

b. Waves of displaced period.

From the expression for y_m we deduce, by putting $\xi = 0$,

$$x_m = \sum_{r=1}^{\infty} \frac{A_r \cos (k-m+2)\frac{r\pi}{k+1} \sin \mu_r \frac{l}{k+1}}{\sin \frac{r\pi}{k+1}}$$

$$= \sum_{r=1}^{\infty} \xi_{m,r}.$$

First, consider the wave-lengths of these oscillations of type $\xi_{m,r}$. Begin with the fundamental:

$$\xi_{m,1} = \frac{A_1 \cos (k-m+2)\frac{\pi}{k+1} \sin \mu_1 \frac{l}{k+1}}{\sin \frac{\pi}{k+1}}.$$

As m increases from 1 to $k+2$ the angle $(k-m+2)\pi/(k+1)$ diminishes from π to 0. In Fig. 4, the line L_1L_1 represents the length of the loop, L_{k+2} is the middle point of it. Points $1, 2, 3, \cdots, k+2$ mark the position of the

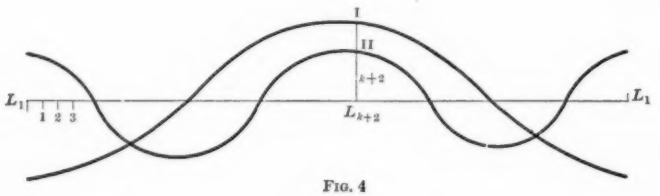

FIG. 4

reactance points. At these points measure off abscissæ equal to $\xi_{1,1}\ \xi_{2,1}\ \xi_{3,1} \cdots$. The extremities of these abscissæ will be on the harmonic curve I. This curve represents the fundamental oscillation of displaced period. Its wave-length is $2l$, the same as the wave-length of the fundamental oscillation before the introduction of the reactance points.

The next component is

$$\xi_{m,2} = \frac{A_2 \cos (k-m+2)\frac{2\pi}{k+1} \sin \mu_2 \frac{l}{k+1}}{\sin \frac{2\pi}{k+1}}.$$

As m increases from 1 to $k+2$ the angle $(k-m+2)2\pi/(k+1)$ diminishes from 2π to 0. It will therefore have four maxima and four zero points on the loop. Its wave-length is l. This component is represented by curve II in Fig. 4.

Take now the last component but one of the first group of free oscillations of displaced period. It is

$$\xi_{m,k} = \frac{A_k \cos (k - m + 2) \frac{k\pi}{k+1} \sin \frac{\mu_k l}{k+1}}{\sin \frac{k\pi}{k+1}}.$$

As m increases from 1 to $k + 2$, the angle $(k - m + 2)k\pi/(k + 1)$ diminishes from $k\pi$ to 0. This component has therefore $2k$ maxima and $2k$ zero points on the loop. Its average wave-length is $2l/k$. The last component of this group is the same as the lowest oscillation of normal period, that is, η_{k+1}.

The components of the first group have therefore the same average wave-lengths as the components of the first group of free oscillations on the loop before the introduction of the reactance points.

It can now be easily shown that this is true for the components of all the other groups and attention should be called here to the fact that the wave-lengths of the free oscillations of displaced periods can be considerably smaller than the distance between two consecutive reactance points. The introduction of the reactance points changes, therefore, neither the number of free oscillations nor their average wave-length. Their periods, however, and their damping constants will be changed as will be shown presently.

c. *Displacement of periods.*

The periods and damping constants of the oscillations of displaced periods are determined from the period equation:

$$\frac{h_r \sin \frac{\mu_r l}{k+1}}{\mu_r} + 2 \cos \frac{\mu_r l}{k+1} = 2 \cos \frac{r\pi}{k+1}.$$

The unknown quantity ν_r, contained in h_r and μ_r, is, of course, a complex quantity, that is, we can write

$$\nu_r = n_r + ip_r.$$

I am not as yet quite ready to discuss fully the general properties of the roots of the period equation. Besides, such a discussion would go considerably beyond the limits of this paper, the principal object of which is to determine the conditions under which the non-uniform conductor of Fig. 3 is equivalent to a uniform loop. I shall limit myself, therefore, to a brief statement of those properties, only, of this equation which bear directly upon these conditions.

First, since

$$\cos \frac{r\pi}{k+1} = \cos \frac{[2s(k+1) \pm r]\pi}{k+1},$$

it follows that the periods and the damping constants of the wave-lengths

$$\frac{2l}{r}, \frac{2l}{2(k+1)\pm r}, \frac{2l}{4(k+1)\pm r}, \cdots, \frac{2l}{2s(k+1)\pm r}, \cdots,$$

are determined from one and the same angle $\psi = r\pi/(k+1)$. This relation between these wave-lengths is illustrated graphically in Fig. 5. The horizontal top bar marked $1, 2, \cdots, k+1, \cdots, 2k+1, 2k+2, 2k+3, \cdots$ contains a numerical record of all the possible wave-lengths

$$\frac{2l}{1}, \frac{2l}{2}, \cdots, \frac{2l}{k+1}, \cdots, \frac{2l}{2k+1}, \cdots, \frac{2l}{2k+2}, \frac{2l}{2k+3}, \cdots.$$

The first vertical column marked $0, 1, 2, 3, \cdots, k+1, \cdots$ contains the record of all possible values of r. The meaning of the black dots is now easily

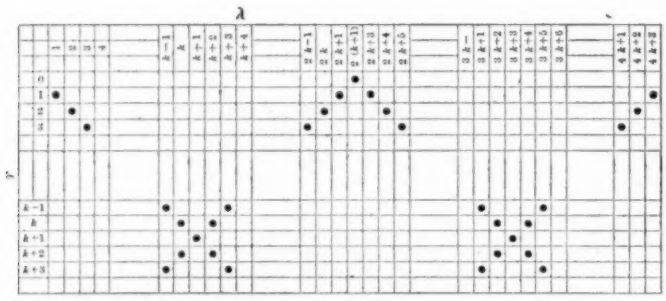

FIG. 5

explained. Take for instance the row of black dots in the horizontal bar 3. These dots are in the columns $3, 2k-1, 2k+5, 4k+1, 4k+7, \cdots$. This means that if, in the period equation:

$$\frac{h}{\mu}\sin\frac{\mu l}{k+1} + 2\cos\frac{\mu l}{k+1} = 2\cos\frac{r\pi}{k+1},$$

we put $r=3$, then the roots of the equation will give us the periods and the damping constants for the wave-lengths

$$\frac{2l}{3}, \frac{2l}{2k-1}, \frac{2l}{2k+5}, \cdots.$$

For this reason these may be called the *concomitant wave-lengths*. An inspection of this table shows that the concomitant wave-lengths are symmetrically arranged with respect to the wave-lengths of normal periods

$$\frac{2l}{k+1}, \quad \frac{2l}{2(k+1)}, \quad \frac{2l}{4(k+1)}, \quad \cdots.$$

Of a given series of concomitant wave-lengths there is one in each of the groups mentioned above. The first one of these groups contains the longest wave-lengths; it may be called the *fundamental group*. The wave-lengths of this group together with their concomitant wave-lengths give all the possible wave-lengths. If, in the period equation, the values $0, 1, 2, \cdots, k+1$ are successively given to r, the roots of the resulting $k+2$ equations will give the periods and damping constants not only for the wave-lengths of the fundamental group but also their concomitants, that is, for all the wave-lengths.

It should be observed that the more nearly $\cos r\pi/(k+1)$ approaches the values $+1$ or -1, the more closely will the concomitant wave-lengths approach the wave-lengths of normal period around which they are symmetrically grouped, and therefore, the more closely will their periods approach the period of the central wave-length. For instance, the wave-lengths

$$\frac{2l}{2k+1}, \quad \frac{2l}{2k+3}, \quad \frac{2l}{4k+3}, \quad \frac{2l}{4k+5}, \quad \cdots$$

are concomitants of the wave-lengths $2l/1$. Their periods are obtained from the period equation by putting $r=1$. This makes $\cos r\pi/(k+1)$ approach the value $+1$ and therefore, the periods of wave-lengths

$$\frac{2l}{2k+1} \text{ and } \frac{2l}{2k+3}, \quad \frac{2l}{4k+3} \text{ and } \frac{2l}{4k+5}, \quad \cdots$$

will approach the periods of the wave-lengths

$$\frac{2l}{2(k+1)}, \quad \frac{2l}{4(k+1)}, \quad \cdots$$

which have been calculated above. When, therefore, k is large, the roots of the period equation:

$$\frac{h}{\mu} \sin \frac{\mu l}{k+1} + 2\cos \frac{\mu l}{k+1} = 2\cos \frac{r\pi}{k+1},$$

come out in pairs of nearly equal magnitude for all values of r which are small in comparison to k. This peculiar property of the roots of the period equation suggests a striking resemblance between the oscillations of a periodically loaded conductor and the luminous vibrations of an incandescent gaseous substance which, in my opinion, deserves serious attention. I expect to discuss this matter more fully on some future occasion.

d. *Equivalence of a non-uniform conductor of the second type to its corresponding uniform conductor.*

Suppose that we now increase the number of reactance points without increasing the total reactance and resistance introduced into the line. In the expression:

$$h = C(\nu^2\rho + \nu R),$$

put

$$\rho = \frac{l\lambda'}{k+1}, \quad R = \frac{lR'}{k+1};$$

then

$$h = \frac{l}{k+1} C(\nu^2\lambda' + \nu R') = \frac{l}{k+1} h'.$$

As the number of reactance points is increased, λ' and R' are to be kept constant; hence ρ and R must vary inversely as $k+1$. When k becomes infinite, all the wave-lengths different from zero will be in the first group of Fig. 5 between the columns 1 and $k+1$. The other columns of this table lose their physical meaning. To show what becomes of the period equation when k approaches the limit ∞, write the period equation in the following form:

$$\frac{\frac{l}{k+1} h' \sin\frac{\mu l}{k+1}}{\mu} - 4\sin^2\tfrac{1}{2}\frac{l\mu}{k+1} = -4\sin^2\tfrac{1}{2}\frac{r\pi}{k+1}.$$

Since wave-lengths different from zero are to be considered, only finite values of r need be considered. Hence when $k = \infty$ the period equation becomes:

$$l^2(h' - \mu^2) = -r^2\pi^2,$$

or

$$C\{\nu^2\lambda' + \nu R' + \nu^2 L + \nu R\} = -\frac{r^2\pi^2}{l^2},$$

or

$$C\{\nu^2(\lambda' + L) + \nu(R' + R)\} = -\frac{r^2\pi^2}{l^2}.$$

This equation is, as it ought to be, the period equation of a uniform conductor of inductance $\lambda' + L$, resistance $R' + R$, and capacity C per unit length. This conductor is the so-called corresponding uniform conductor of the periodically loaded loop.

When k is large but not infinite the same relation will exist approximately, and it is proposed to ascertain now the degree of this approximation.

We have seen that $l\mu/(k+1)$ varies with r as follows:

$$\frac{l\mu}{k+1} = s\pi, \quad \text{where} \quad r = s(k+1),$$

s being any integer from 0 to ∞. Thus when

$$\frac{r\pi}{k+1} \quad \text{is} \quad 0, \pi, 2\pi, 3\pi, 4\pi, \cdots,$$

$$\frac{\mu l}{k+1} \quad \text{will be} \quad 0, \pi, 2\pi, 3\pi, 4\pi, \cdots.$$

Since $\mu l/(k+1)$ varies continuously with r it follows that when $r\pi/(k+1)$ is small, $\mu l/(k+1)$ will also be small since both vanish simultaneously. It is well to introduce here the physical meaning of $r/(k+1)$. Let λ_r be the wave-length of the wave for which $\psi = r\pi/2(k+1)$; then $\lambda_r = 2l/r$, and therefore

$$\frac{r\pi}{k+1} = \frac{2\pi}{\lambda_r}\frac{l}{k+1} = 2\pi\frac{l_1}{\lambda_r},$$

where l_1 is the distance between two consecutive coils. Let 2π be the angular distance of λ_r, and θ_r the angular distance of l_1 corresponding to the period of λ_r; then

$$\frac{l_1}{\lambda_r} = \frac{\theta_r}{2\pi}.$$

Therefore

$$\frac{2\pi}{k+1} = \theta_r.$$

Since for small values of $2\pi/(k+1)$ the angle $\mu l/(k+1)$ is nearly equal to $r\pi/(k+1)$ it follows that within these limits $\mu l/(k+1)$ is approximately equal to the angle θ_r. It follows therefore that wave-lengths for which θ_r is sufficiently small, the period equation:

$$\frac{l}{k+1}\cdot\frac{h'}{\mu}\sin\frac{\mu l}{k+1} - 4\sin^2\tfrac{1}{2}\frac{l\mu}{k+1} = -4\sin^2\tfrac{1}{2}\frac{r\pi}{k+1},$$

can be written:

$$\left(\frac{l}{k+1}\right)^2 h' - \left(\frac{l}{k+1}\right)^2\mu^2 = -\left(\frac{r\pi}{k+1}\right)^2,$$

or

$$C\{\nu_r^2(\lambda' + L) + \nu_r(R' + R)\} = -\left(\frac{r\pi}{k+1}\right)^2;$$

that is, the periodically loaded conductor has approximately the same period, damping constant, and, of course, the same wave-length as its corresponding uniform conductor. The degree of approximation is of the same order as the degree of approximation between one half of the angular distance separating two consecutive coils and the sine of that distance.

It should be observed that if the whole loop is divided into a number of equal parts and the consecutive parts are connected to each other by mutual induction, then the non-uniform conductor thus obtained will act in the same way as the non-uniform conductor of Fig. 3 of this section. Another arrangement which is equivalent to the two arrangements so far described is obtained by placing bridges at periodic intervals of the uniform loop, each bridge consisting of a coil of proper resistance and inductance, and the distance between the bridges being adjusted in accordance with the general rule formulated here. The mathematical analysis of these two arrangements does not differ essentially from the one given here and can be easily worked out.

A rule governing the degree of approximation between a non-uniform conductor and its corresponding uniform conductor has thus been established, and it has been proved that the same rule is applicable to both types of non-uniform conductors and also to both forced and free oscillations. The principal object of this investigation has, therefore, been accomplished.

COLUMBIA UNIVERSITY.

UEBER SYSTEME VON DIFFERENTIALGLEICHUNGEN DENEN VIERFACH PERIODISCHE FUNCTIONEN GENÜGE LEISTEN*

VON

MARTIN KRAUSE

In der Theorie der elliptischen Functionen haben sich die linearen Differentialgleichungen von Bedeutung gezeigt, deren Coefficienten gewöhnliche doppelt periodische Functionen sind, während die Integrale sich als doppelt periodische Functionen zweiter Art darstellen lassen. Insbesonders sind zwei Gleichungen in einer Reihe von Arbeiten untersucht worden, die LAMÉsche † und die PICARDsche ‡ Differentialgleichung zweiter Ordnung. Nachdem Verfasser § versucht hat, im Gebiete der Systeme von Differentialgleichungen zweiter Ordnung, deren Coefficienten vierfach periodische Functionen sind, diejenigen herauszugreifen, welche als einfachstes Analagon zu der LAMÉschen Differentialgleichung angesehen werden können, sollen im Folgenden Systeme von Differentialgleichungen aufgestellt werden, welche als Analogon zu der PICARDschen Differentialgleichung zweiter Ordnung gelten können. Die Anschauungen, welche Verfasser zu seinen Resultaten geführt haben, beruhen auf den Arbeiten von Herrn FUCHS, insbesondere derjenigen, welche sich in den Nachrichten der Göttinger Gesellschaft der Wissenschaften vom Jahre 1878 und im 81sten Bande des Crelleschen Journals vorfinden.

§ 1.

Als Ausgangspunkt der Theorie wählen wir die Function:

$$Z = G(z)^{\frac{1}{2}} e^{c\int \frac{z^{\nu+1}\,dz}{G(z)\sqrt{R(z)}}}, \tag{1}$$

wobei $G(z)$ eine ganze Function von z aus lauter ungleichen Factoren bedeutet, welche die Form hat:

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† Siehe hierüber vor allem HERMITE: *Sur quelques applications des fonctions elliptiques.* Paris 1885.

‡ PICARD: *Sur quelques applications de la théorie des fonctions elliptiques*, Comptes Rendus, Band 89.

§ *Sur les systèmes d'équations différentielles auxquels satisfont les fonctions quadruplement periodiques de seconde espèce*, Comptes Rendus, 1898.

Ueber verallgemeinerte Lamé–Hermitesche Differentialgleichungen für den Fall zweier veränderlichen Grössen, Berichte der Leipziger Gesellschaft der Wissenschaften, 1898.

$$G(z) = (z - t_1)(z - t_2) \cdots (z - t_m) = \sum_{s=0}^{s=m} p_s z^{m-s},$$

während unter $R(z)$ der Ausdruck verstanden wird:

$$R(z) = z(1 - z)(1 - \kappa^2 z)(1 - \lambda^2 z)(1 - \mu^2 z) = \sum_{s=0}^{s=5} r_s z^{5-s}.$$

Diese Function Z leistet dann der Differentialgleichung Genüge:

$$(2) \qquad R(z)\frac{d^2Z}{dz^2} + \left(\tfrac{1}{2}R'(z) - \frac{(\nu+1)R(z)}{z}\right)\frac{dZ}{dz} = \frac{Z}{4G}M,$$

$$M = G'(z)R'(z) - \frac{2(\nu+1)G'(z)R(z)}{z} + 2G''(z)R(z) + \frac{4c^2 z^{2\nu+2} - R(z)G'(z)^2}{G(z)}.$$

Wir wollen nun die Bedingung hinzunehmen, dass der Coefficient von Z vom Unendlichkeitspunkt abgesehen den Nullpunkt als einzigen singulären Punkt besitzen soll, dann muss jedenfalls die Grösse:

$$G'(z)^2R(z) - 4c^2 z^{2\nu+2}$$

durch $G(z)$ theilbar sein, oder also die Form haben:

$$z^{2\nu+2}\, G(z)\, \phi(z),$$

wobei dann $z^{2\nu+2}\,\phi(z)$ eine ganze Function von z bedeutet. Nehmen wir an, dass

$$2\nu + 2 < 2m + 3$$

ist, so wird der Grad von $z^{2\nu+2}\,\phi(z)$ gleich $m + 3$ sein, so zwar, dass gesetzt werden kann:

$$(3) \qquad \phi(z) = \frac{c_{-2\nu-1}}{z^{2\nu+1}} + \cdots + c_{m-2\nu+1} z^{m-2\nu+1}.$$

Es ergiebt sich dann die Beziehung:

$$\frac{G'(z)^2R(z)}{z^{2\nu+2}} - 4c^2 = G(z)\phi(z).$$

Wir differentiiren links und rechts nach z. Der Differentialquotient auf der linken Seite nimmt die Form an:

$$G'(z)\phi_1(z),$$

während er auf der rechten Seite wird:

$$G(z)\phi'(z) + G'(z)\phi(z).$$

Da $G(z)$ und $G'(z)$ keinen gemeinsamen Factor besitzen, so muss eine Gleichung von der folgenden Form bestehen:

$$z^{2\nu+2}\phi'(z) = G'(z)(Az^3 + Bz^2 + Cz + D), \tag{4}$$

oder also wir erhalten die Beziehung:

$$z^{2\nu+2}\phi_1(z) = G(z)(Az^3 + Bz^2 + Cz + D) + z^{2\nu+2}\phi(z). \tag{5}$$

Andrerseits ergiebt sich, wie aus der Definition von $\phi_1(z)$ folgt, die Gleichung:

$$z^{2\nu+2}\phi_1(z) = 2G''(z)R(z) - G'(z)\{2\nu + 1 + 2\nu r_3 z + (2\nu-1)r_2 z^2 + (2\nu-2)r_1 z^3 + (2\nu-3)r_0 z^4\}.$$

Aus dieser Gleichung folgt für $\phi_1(z)$ die Form:

$$\phi_1(z) = \sum_{s=-3}^{s=m} q_s z^{m-2\nu-s-2}, \tag{6}$$

wobei q_s den Werth besitzt:

$$q_s = \sum_{\epsilon=-1}^{\epsilon=3} (m-s-\epsilon)(2m-2\nu-2s-\epsilon-2)p_{s+\epsilon}r_{3-\epsilon}. \tag{7}$$

Differentiiren wir in Gleichung (5) beide Seiten nach z und berücksichtigen Gleichung (4), so erhalten wir die Beziehung:

$$\phi_1'(z) = \sum q_s' z^{m-2\nu-s-3}, \tag{8}$$

wobei gesetzt ist:

$$q_s' = (2m-2s-2\nu-5)p_{s+3}A + (2m-2s-2\nu-4)p_{s+2}B + (2m-2s-2\nu-3)p_{s+1}C + (2m-2s-2\nu-2)p_sD. \tag{9}$$

Aus Gleichung (6) ergiebt sich aber für $\phi_1'(z)$ ein zweiter Ausdruck, nämlich:

$$\phi_1'(z) = \sum (m-2\nu-s-2)q_s z^{m-2\nu-s-3}. \tag{10}$$

Die Vergleichung der beiden für $\phi_1'(z)$ gefundenen Werthe führt dann zu der Recursionsformel:

$$(m-2\nu-s-2)q_s = q_s'. \tag{11}$$

Sondern wir die Werthe $m = \nu$ und $m = \nu+1$ ab und setzen für s der Reihe nach: $-3, -2, -1, 0$, so ergeben sich für die Grössen A, B, C, D die Werthe:

$$A = m(m-2\nu+1)r_0,$$

$$B = m(m-2\nu)r_1 - (2m-2\nu-1)p_1r_0, \tag{12}$$

$$C = m(m-2\nu-1)r_2 - 2(m-\nu-1)p_1(r_1 - p_1r_0) - 2(2m-2\nu-3)p_2r_0,$$

$$D = m(m-2\nu-2)r_3 - (2m-2\nu-3)p_1(r_2 - p_1r_1 + p_1^2r_0) - 2(2m-2\nu-4)p_2r_1 + (6m-6\nu-11)p_1p_2r_0 - 3(2m-2\nu-5)p_3r_0.$$

Aus den folgenden Gleichungen für $s = 1, 2, \cdots, m$ sind die Grössen p mit Ausnahme einer einzigen bestimmt, welche willkührlich bleibt.

Die Differentialgleichung (2) nimmt die Form an:

$$(13)\quad R(z)\frac{d^2Z}{dz^2} + \left(\tfrac{1}{2}R'(z) - \frac{(\nu+1)R(z)}{z}\right)\frac{dZ}{dz} = \frac{Z}{4}(Az^3 + Bz^2 + Cz + D).$$

Die beiden Integrale besitzen die Form:

$$G(z)^{\frac{1}{2}} e^{\pm c\int \frac{z^{\nu+1}dz}{G(z)\sqrt{R(z)}}},$$

wobei das positive Zeichen zu dem einen, das negative zu dem anderen Integral gehört.

§ 2.

Wir haben bei den Erörterungen des vorigen Paragraphen die Ungleichheit gebraucht:

$$2m + 3 > 2\nu + 2.$$

Im Uebrigen kann m eine jede ganze positive Zahl bedeuten. Die Resultate vereinfachen sich etwas, wenn m so bestimmt wird, dass

$$A = 0$$

wird. Es geschieht das für

$$m = 2\nu - 1.$$

Mit diesem Werth von m wollen wir weiter rechnen, bemerken aber, dass auch der allgemeine Fall im Prinzip genau so behandelt werden kann, wie der soeben angeführte specielle.

Wir wollen uns dann die Gleichung (13) der vorigen Paragraphen für zwei unabhängige Veränderliche gebildet denken, so dass wir die Gleichungen erhalten;

$$(1)\quad \begin{aligned} R(z_1)\frac{d^2Z}{dz_1^2} + \left(\tfrac{1}{2}R'(z_1) - \frac{(\nu+1)R(z_1)}{z_1}\right)\frac{dZ}{dz_1} &= \frac{Z}{4}(Bz_1^2 + Cz_1 + D), \\ R(z_2)\frac{d^2Z}{dz_2^2} + \left(\tfrac{1}{2}R'(z_2) - \frac{(\nu+1)R(z_2)}{z_2}\right)\frac{dZ}{dz_2} &= \frac{Z}{4}(Bz_2^2 + Cz_2 + D). \end{aligned}$$

Die Integrale der ersten haben die Form:

$$Z_1^{(1)} = G(z_1)^{\frac{1}{2}} e^{c\int \frac{z_1^{\nu+1}dz_1}{G(z_1)\sqrt{R(z_1)}}}, \qquad Z_1^{(2)} = G(z_1)^{\frac{1}{2}} e^{-c\int \frac{z_1^{\nu+1}dz_1}{G(z_1)\sqrt{R(z_1)}}};$$

die Integrale der zweiten können wir schreiben:

$$Z_2^{(1)} = G(z_2)^{\frac{1}{2}} e^{-c\int \frac{z_2^{\nu+1}dz_2}{G(z_2)\sqrt{R(z_2)}}}, \qquad Z_2^{(2)} = G(z_2)^{\frac{1}{2}} e^{c\int \frac{z_2^{\nu+1}dz_2}{G(z_2)\sqrt{R(z_2)}}}.$$

Die Functionen:

$$Z^{(1)} = Z_1^{(1)} Z_2^{(1)} \text{ und } Z^{(2)} = Z_1^{(2)} Z_2^{(2)}$$

genügen dann jedenfalls beiden Differentialgleichungen.

Wir führen jetzt die hyperelliptischen Functionen nach ROSENHAIN ein, indem wir setzen:

$$(2)\qquad \begin{aligned} 2du_1 &= \frac{dz_1}{\sqrt{R(z_1)}} - \frac{dz_2}{\sqrt{R(z_2)}}, \\ 2du_2 &= -\frac{z_1\,dz_1}{\sqrt{R(z_1)}} + \frac{z_2\,dz_2}{\sqrt{R(z_2)}}. \end{aligned}$$

Die Grössen $Z^{(1)}$ und $Z^{(2)}$ können dann auch als Functionen von u_1 und u_2 angesehen werden. Um das anzudeuten, wollen wir an Stelle des Buchstabens Z uns den Buchstaben ϕ eingeführt denken. Die Differentialgleichungen (1) nehmen dann die Form an:

$$\begin{aligned} &\frac{\partial^2\phi}{\partial u_1^2} - 2z_1 \frac{\partial^2\phi}{\partial u_1 \partial u_2} + z_1^2 \frac{\partial^2\phi}{\partial u_2^2} - \frac{2(\nu+1)}{z_1}\sqrt{R(z_1)}\,\frac{\partial\phi}{\partial u_1} + 2\nu\sqrt{R(z_1)}\,\frac{\partial\phi}{\partial u_2} \\ &\qquad = (Bz_1^2 + Cz_1 + D)\phi, \\ &\frac{\partial^2\phi}{\partial u_1^2} - 2z_2 \frac{\partial^2\phi}{\partial u_1 \partial u_2} + z_2^2 \frac{\partial^2\phi}{\partial u_2^2} + \frac{2(\nu+1)}{z_2}\sqrt{R(z_2)}\,\frac{\partial\phi}{\partial u_1} - 2\nu\sqrt{R(z_2)}\,\frac{\partial\phi}{\partial u_2} \\ &\qquad = (Bz_2^2 + Cz_2 + D)\phi. \end{aligned}$$

Aus ihnen ergeben sich durch einfache Operationen die beiden Differentialgleichungen:

$$(3)\qquad \begin{aligned} &\frac{\partial^2\phi}{\partial u_1^2} - s_2 \frac{\partial^2\phi}{\partial u_2^2} - \frac{(\nu+1)}{s_2\kappa\lambda\mu}\,\frac{\partial}{\partial u_1}\frac{\vartheta_1^2(v)}{\vartheta_0^2(v)}\,\frac{\partial\phi}{\partial u_1} + \frac{\nu}{\kappa\lambda\mu}\,\frac{\partial}{\partial u_2}\frac{\vartheta_1^2(v)}{\vartheta_0^2(v)}\,\frac{\partial\phi}{\partial u_2} \\ &\qquad = \phi(-Bs_2 + D), \\ &s_1\frac{\partial^2\phi}{\partial u_1^2} - 2s_2 \frac{\partial^2\phi}{\partial u_1 \partial u_2} - \frac{(\nu+1)}{s_2\kappa\lambda\mu}\left(s_1\frac{\partial}{\partial u_1}\frac{\vartheta_1^2(v)}{\vartheta_0^2(v)} - s_2\frac{\partial}{\partial u_2}\frac{\vartheta_1^2(v)}{\vartheta_0^2(v)}\right)\frac{\partial\phi}{\partial u_1} \\ &\qquad + \frac{\nu}{\kappa\lambda\mu}\,\frac{\partial}{\partial u_1}\frac{\vartheta_1^2(v)}{\vartheta_0^2(v)}\,\frac{\partial\phi}{\partial u_2} = \phi(Cs_2 + Ds_1). \end{aligned}$$

In diesen Gleichungen ist gesetzt:

$$s_1 = \frac{1}{\kappa^2} + \frac{1}{\kappa^2\lambda^2\mu^2}\left(\kappa^4\frac{\theta_1'(u_1)_0^2}{\vartheta_0^2}\frac{\vartheta_1^2(v)}{\vartheta_0^2(v)} + \frac{\theta_3'(u_2)_0^2}{\vartheta_0^2}\frac{\vartheta_3^2(v)}{\vartheta_0^2(v)}\right), \quad s_2 = \frac{1}{\kappa\lambda\mu}\frac{\vartheta_1^2(v)}{\vartheta_0^2(v)},$$

$$\theta_\alpha'(u_\epsilon)_0 = \frac{\partial}{\partial u_\epsilon}\,\vartheta_\alpha(v_1, v_2)_{(v_1, v_2) = (0, 0)},$$

während v_1 und v_2 die Argumente der Thetafunctionen bedeuten, welche den Argumenten u_1 und u_2 der hyperelliptischen Functionen entsprechen und mit ihnen durch die bekannten linearen Relationen verbunden sind.

Es sind das zwei Differentialgleichungen mit vierfach periodischen Coefficienten, welche wir als Analogon zu der schon citirten Picardschen Differentialgleichung zweiter Ordnung im Gebiete der Thetafunctionen zweier Veränderlichen ansehen können und zwar für den ersten Fall von m.

Die Darstellung zweier Integrale dieser Gleichungen durch Thetafunctionen kann ähnlich durchgeführt werden, wie es in den citirten Arbeiten des Verfassers für die verallgemeinerten LAMÉ-HERMITEschen Differentialgleichungen geschehen ist. Entwickeln wir:

$$\frac{z_1^{\nu+1}}{G(z_1)} \quad \text{und} \quad \frac{z_2^{\nu+1}}{G(z_2)}$$

in Partialbrüche und berücksichtigen die angegebenen Werthe von c, so gelangen wir zu Integralen derselben Form, wie sie an den angegebenen Orten durch Thetafunctionen dargestellt worden sind.

Wir müssen dazu den einzelnen Wurzeln der Gleichung:

$$G(z) = 0\,,$$

d. h. den Grössen t_s Parameter $a_1^{(s)}$, $a_2^{(s)}$ zuordnen, zwischen denen einerseits die Gleichung besteht:

$$\vartheta_3(a_1^{(s)}, a_2^{(s)}) = 0\,,$$

während sie andrerseits den Gleichungen Genüge leisten:

$$\frac{\vartheta_1^2(a_1^{(s)}, a_2^{(s)})}{\vartheta_0^2(a_1^{(s)}, a_2^{(s)})} = \frac{\lambda\mu}{\kappa}\, t_s\,, \quad \ldots\ldots\,.$$

Unter solchen Umständen können wir die folgenden Sätze aussprechen:

I. *Die Differentialgleichungen* (3) *ändern sich nicht, wenn an Stelle von* u_1, u_2 *resp. gesetzt wird:* $-u_1, -u_2$.

II. *Die Coefficienten ihrer linken Seiten sind unabhängig von den Grössen* $a_1^{(s)}$, $a_2^{(s)}$.

III. *Zwei ihrer Integrale haben die Form:*

$$\phi^{(1)} = \prod_{s=1}^{s=m} \frac{\vartheta_1(v_1 + a_1^{(s)}, v_2 + a_2^{(s)})}{\vartheta_0(v_1, v_2)}\, e^{\lambda_s}\,,$$

$$\phi^{(2)} = \prod_{s=1}^{s=m} \frac{\vartheta_1(v_1 - a_1^{(s)}, v_2 - a_2^{(s)})}{\vartheta_0(v_1, v_2)}\, e^{-\lambda_s}\,,$$

wobei unter Fortlassung des Index s gesetzt ist:

$$\lambda = -\sum_{\epsilon=1}^{\epsilon=2} \frac{\partial}{\partial a_\epsilon} \log \vartheta_1(a_1, a_2) u_\epsilon - \frac{\kappa}{2\lambda\mu} \frac{\partial}{\partial a_2} \frac{\vartheta_1^2(a_1, a_2)}{\vartheta_0^2(a_1, a_2)}\, u_1\,.$$

Auch die noch fehlenden beiden Integrale können durch Thetafunctionen dargestellt werden—es müssen dazu aber andere Argumente, als die bisherigen eingeführt werden.

ON LINEAR CRITERIA FOR THE DETERMINATION OF THE RADIUS OF CONVERGENCE OF A POWER SERIES*

BY

EDWARD B. VAN VLECK

The most common criteria for the determination of the radius of convergence of a power series $A_0 + A_1x + A_2x^2 + \cdots$ are based upon well known convergence-theorems of CAUCHY and involve expressions containing at most two coefficients of the series. According to one criterion the radius of convergence is the limit of $|A_n^{-1/n}|$ or, in case no limit exists, the reciprocal of the greatest affix of any point of condensation of the sequence $|A_n^{1/n}|$, $(n = 1, 2, 3, \cdots)$. CAUCHY'S ratio test, on the other hand, gives the radius of convergence only when $|A_n/A_{n-1}|$ approaches a limit, the reciprocal of which is then the desired radius.† But though the first criterion is perfectly general and of the highest value for theoretical investigations, it nevertheless has little superiority as a practical test for the determination of the radius of convergence of a given series. For when $|A_n/A_{n-1}|$ approaches a limit, the two criteria are of equal import, and the ratio-test is commonly used, being easier of application. On the other hand, in most cases when no limit for the ratio exists, the determination of the limit of $|A_n^{1/n}|$, or of its points of condensation, is extremely difficult,‡ as, for example, when more than two coefficients are connected in the law governing their formation.

The object of this paper is to establish certain criteria for the convergence of a power series when the $(n+1)$th coefficient A_n is connected with the preceding coefficients by a linear relation which tends to take a limiting form as n increases indefinitely. The criteria obtained include CAUCHY'S ratio test as a special case and may, indeed, be regarded as an extension of that test. They are applicable in many cases in which the simple ratio-test fails. One special class of series to which they apply (see sections 7 and 8) has been previously considered by

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† When the ratio fails to approach a limiting value, only a minor limit to the radius of convergence is obtained from the sequence $|A_{n-1}/A_n|$.

‡ An exception worthy of note is a series in which A_n does not approach a limiting value but has a finite upper limit. The radius of convergence is then equal to unity.

HADAMARD* and includes every series defining a function which has only polar singularities on the circle of convergence of the series. Another application, which will be made in a subsequent paper, is to the subject of linear differential equations.

The linear criteria can also be extended to power series in two or more variables. The last two theorems of the present paper give the extension for the case of two variables, and may be of some interest, inasmuch as criteria for the convergence of power series in two or more variables seem as yet to be very rare. †

1. *Simple proof of convergence.*

Consider a series:

$$S = A_0 + A_1 x + A_2 x^2 + \cdots, \tag{1}$$

in which after some fixed term every $p+1$ consecutive coefficients are connected by a linear relation:

$$A_n = b_1^{(n)} A_{n-1} + b_2^{(n)} A_{n-2} + \cdots + b_p^{(n)} A_{n-p}. \tag{2}$$

If the coefficients $b_i^{(n)}$ in this relation, as n indefinitely increases, converge toward fixed limits b_i, the series must converge for values of x other than zero. A simple proof of this is as follows. According to the hypothesis of the existence of a limiting form for (2), if we arbitrarily prescribe a small positive quantity ϵ, an integer m can be found such that for $n \geqq m$ the moduli of the p differences $b_i^{(n)} - b_i$ will all be less than ϵ. Denote now with b' the greatest of the quantities $|b_i| + \epsilon$ and with A' the greatest of the moduli $|A_{m-p}|$, $|A_{m-p+1}|$, $\cdots$, $|A_{m-1}|$. Then since for $n \geqq m$ every $|b_i^{(n)}| < b'$, it follows first from (2), by putting $n = m$, that

$$|A_m| < pb'A'.$$

Hence if c is the larger of the two quantities pb' and 1, none of the p moduli from $|A_{m-p+1}|$ to $|A_m|$ inclusive can exceed cA'. Placing next $n = m+1$ in (2), we find that

$$|A_{m+1}| < pb'cA' \leqq c^2 A'$$

and, continuing in the same manner, we have in general

$$|A_{m+i}| < c^{i+1} A'.$$

Therefore after the first m terms of (1) the moduli of the successive terms will be less than the corresponding terms of

$$A'cx^m + A'c^2x^{m+1} + A'c^3x^{m+2} + \cdots.$$

*Liouville's Journal, ser. 4, vol. 8.

† In a dissertation published in volume 34 of the Mathematische Annalen J. HORN has given criteria of an entirely different character for the convergence of certain series which he terms hypergeometric series in two or more variables.

The latter series obviously converges for $|x| < 1/c$. It follows that S must converge within a circle of radius $1/c$ having its center at the origin.

2. *On the formation of the coefficients.*

For many purposes a simple proof such as this of the convergence of S gives all that is desired, although only a minor limit for the radius of convergence has been found. The proof, for example, suffices as an existence-theorem in certain simple cases in the theory of linear differential equations. To determine, however, what the radius of convergence is, another line of proof must be adopted. Consider as a preliminary the power series $a_0 + a_1x + a_2x^2 + \cdots$ for

$$\frac{1 + c_1x + \cdots + c_qx^q}{1 - b_1x - \cdots - b_px^p},$$

which may be obtained by the method of indeterminate coefficients. Beginning with the $(q+2)$th term of the series, each coefficient is obtained from the preceding coefficients by the recurring relation:

$$a_n = b_1a_{n-1} + b_2a_{n-2} + \cdots + b_pa_{n-p}. \tag{3}$$

Conversely, if, after some fixed point in a series, the coefficients are connected by the relation (3), the series is the expansion of some rational fraction $G(x)/B_p(x)$ whose denominator $B_p(x)$ is the polynomial $1 - b_1x - \cdots - b_px^p$. A particular case which we shall have occasion to use is the series for $1/B_p(x)$, the first few coefficients of which are obtained from the equations:

$$\begin{aligned} a_0 &= 1, \\ a_1 &= b_1a_0, \\ a_2 &= b_1a_1 + b_2a_0, \\ a_3 &= b_1a_2 + b_2a_1 + b_3a_0, \text{ etc.}; \end{aligned} \tag{4}$$

the law for the formation of the coefficients may then be said to hold from the outset. In any case, if the fraction has been expressed in its lowest terms, the radius of convergence of the series is equal to the distance from the origin to the nearest root (or roots) of $B_p(x) = 0$.

Let us now return to our series S. The characteristic relation (2) may be written in the form:

$$A_n - (b_1 + \epsilon_1^{(n)})A_{n-1} - (b_2 + \epsilon_2^{(n)})A_{n-2} - \cdots - (b_p + \epsilon_p^{(n)})A_{n-p} = 0, \tag{5}$$

or

$$A_n = b_1A_{n-1} + b_2A_{n-2} + \cdots + b_pA_{n-p} + \epsilon_n, \tag{6}$$

in which

$$\epsilon_n = \epsilon_1^{(n)}A_{n-1} + \epsilon_2^{(n)}A_{n-2} + \cdots + \epsilon_p^{(n)}A_{n-p}. \tag{7}$$

Beginning with the $(m+1)$th term of the series, let each coefficient be expressed in terms of the p coefficients $A_{m-p}, A_{m-p+1}, \cdots, A_{m-1}$ and of the quantities $\epsilon_m, \epsilon_{m+1}, \cdots$. Let also the part of each coefficient which is independent of the latter quantities be written first, then in turn the parts containing ϵ_m, $\epsilon_{m+1}, \cdots$. Thus, for example,

$$A_m = (\ ) + \epsilon_m,$$
$$A_{m+1} = (\ ) + b_1\epsilon_m + \epsilon_{m+1},$$
$$A_{m+2} = (\ ) + b_1(b_1\epsilon_m) + b_2\epsilon_m + b_1\epsilon_{m+1} + \epsilon_{m+2},$$

in which the parts independent of the ϵ_i are indicated by parentheses. If now, after the mth term, we should neglect all the quantities ϵ_i, $(i \geqq m)$, the series would be the expansion of some rational fraction $G(x)/B_p(x)$. As regards ϵ_m, it will be observed that it enters explicitly into the successive coefficients of S in the combinations:

$$\epsilon_m, \quad b_1\epsilon_m, \quad b_1(b_1\epsilon_m) + b_2\epsilon_m, \quad b_1[b_1(b_1\epsilon_m) + b_2\epsilon_m] + b_2[b_1\epsilon_m] + b_3, \quad \cdots,$$

that is to say, in accordance with the law expressed in (4). If, therefore, the parts of the successive terms of S into which ϵ_m explicitly enters should be collected, they would give a power series $P(x)$ for $\epsilon_m x^m/B_p(x)$. The portion of the series depending upon any subsequent ϵ_n gives in like manner the expansion of $\epsilon_n x^n/B_p(x)$ into a power series. We conclude therefore that *the coefficient of any power of x in the series S can be obtained by summing the coefficients of the same power of x in the various series which represent the respective members of the function:*

$$F(x) = \frac{G(x)}{B_p(x)} + \sum_{n=m}^{\infty} \frac{\epsilon_n x^n}{B_p(x)}. \tag{8}$$

The question whether it is permissible to rearrange S so as to put it into the form (8), or, in other words, whether $F(x)$ and S are identical, will be considered later (§ 5). For the present it suffices that the coefficients of S can be obtained from (8) in the manner stated.

3. *Fundamental proof of convergence.*

We now proceed, with the aid of (8), to prove anew the convergence of S. According to our fundamental hypothesis the relation (5) tends with increasing n to take the form (3). If, therefore, any positive quantity ϵ, however small, be prescribed, an integer m can be found such that for $n \geqq m$ every $|\epsilon_i^{(n)}|$ will be less than ϵ. To obtain a major limit to $|A_n|$ we shall have recourse to the following well known theorem: If a series

$$\sum_{i=0}^{\infty} c_i x^i$$

converges absolutely for $|x| = \xi$, then $|c_n| < \gamma\xi^{-n}$ where γ is some fixed number independent of n. Let ξ designate here any positive quantity smaller than the radius of convergence of the series for $G(x)/B_p(x)$ and for $1/B_p(x)$, and let γ be taken sufficiently large that the inequality shall apply simultaneously to both series. Since the first m coefficients of S are also coefficients of the series for $G(x)/B_p(x)$, we have

$$|A_{m-i}| < \frac{\gamma}{\xi^{m-i}}, \qquad (i = 1, 2, \cdots, m).$$

It follows then from (7), by putting $n = m$, that

$$|\epsilon_m| < \epsilon\gamma\left(\frac{1}{\xi^{m-1}} + \frac{1}{\xi^{m-2}} + \cdots + \frac{1}{\xi^{m-p}}\right) = \frac{X}{\xi^m},$$

where

$$X = \epsilon\gamma(\xi + \xi^2 + \cdots + \xi^p).$$

Hence $|A_m|$, which is the modulus of the coefficient of x^m in the series for

$$\frac{G(x)}{B_p(x)} + \frac{\epsilon_m x^m}{B_p(x)},$$

is less than

$$\frac{\gamma}{\xi^m} + \gamma|\epsilon_m| = \frac{\gamma(1+X)}{\xi^m}.$$

Since also $|A_{m-i}| < \gamma(1+X)\xi^{-(m-i)}$, we next obtain by placing $n = m+1$ in (7)

$$(9) \qquad |\epsilon_{m+1}| < \epsilon\gamma(1+X)\left(\frac{1}{\xi^m} + \frac{1}{\xi^{m-1}} + \cdots + \frac{1}{\xi^{m-p+1}}\right) = \frac{X(1+X)}{\xi^{m+1}}.$$

From this it follows that A_{m+1}, or the coefficient of x^{m+1} in the series for

$$\frac{G(x)}{B_p(x)} + \frac{\epsilon_m x^m}{B_p(x)} + \frac{\epsilon_{m+1}x^{m+1}}{B_p(x)},$$

has a modulus smaller than

$$(10) \qquad \gamma\frac{(1+X)}{\xi^{m+1}} + \gamma\frac{X(1+X)}{\xi^{m+1}} = \gamma\frac{(1+X)^2}{\xi^{m+1}}.$$

To obtain a major limit for $|\epsilon_{m+2}|$ and $|A_{m+2}|$ we have only to replace m by $m+1$ and γ by $\gamma(1+X)$ in (9) and (10). Proceeding in this manner we obtain

$$(11) \qquad |\epsilon_{m+q}| < \frac{X(1+X)^q}{\xi^{m+q}}, \qquad |A_{m+q}| < \frac{\gamma(1+X)^{q+1}}{\xi^{m+q}}.$$

From the last inequality it is evident that S must converge within the circle of convergence of

$$1 + \frac{1+X}{\xi}x + \frac{(1+X)^2}{\xi^2}x^2 + \cdots, \tag{12}$$

the radius of which is $\xi/(1+X)$. But by taking m sufficiently large the value of ϵ and therefore of X may be made smaller than any assigned quantity, while ξ denotes any positive value smaller than the radius of convergence of the series for $1/B_p(x)$. The series S must therefore converge within a circle C whose radius is the distance from the origin to the nearest root (or roots, if several have equal moduli) of $B_p(x) = 0$. The result thus reached may be recapitulated as follows:

I. *Let a series* $A_0 + A_1x + A_2x^2 + \cdots$ *be given in which, after some fixed term, every* $p+1$ *consecutive coefficients are connected by a linear relation:*

$$A_n = (b_1 + \epsilon_1^{(n)})A_{n-1} + (b_2 + \epsilon_2^{(n)})A_{n-2} + \cdots + (b_p + \epsilon_p^{(n)})A_{n-p}.$$

If for any arbitrarily assigned positive value ϵ, *an integer* m *can be found such that for* $n \geqq m$ *every* $|\epsilon_i^{(n)}|$ *will be less than* ϵ, *the series will converge within a circle* C *whose center is the origin and whose radius is the distance from the origin to the nearest root (or roots) of the polynomial:*

$$1 - b_1x - b_2x^2 - \cdots - b_px^p.$$

A special case of this theorem is that in which $p = 1$. The root of the polynomial, $1/b_1$, is then also the limit of A_{n-1}/A_n, and we have CAUCHY'S ratio test for the determination of the circle of convergence.

4. *The linear relation with the smallest number of terms.*

A necessary consequence of the equation (5) is the existence of similar relations containing a greater number of terms. For if in (5) we replace n by $n+1$ and subtract the original equation multiplied with a constant a, we obtain

$$A_{n+1} - (b_1 + a + \epsilon_1^{(n+1)})A_n - (b_2 - ab_1 + \epsilon_2^{(n+1)})A_{n-1} - \cdots - (-ab_p + \epsilon_{p+1}^{(n+1)})A_{n-p} = 0$$

in which the $\epsilon_i^{(n+1)}$ denote certain infinitesimals whose moduli may be made smaller than any assigned quantity by sufficiently increasing n. But this is exactly the form for a linear relation corresponding to the polynomial $(1-ax)B_p(x)$. We conclude therefore that *for every polynomial which contains* $B_p(x)$ *as a factor there exists a linear relation between the coefficients of* S *whose limiting form corresponds to this polynomial.*

Suppose now that there is a linear relation of the same character as (5) but containing a smaller number of terms, say

$$A_r - (b_1' + \epsilon_1^{(r)})A_{r-1} - \cdots - (b_{p'}' + \epsilon_{p'}^{(r)})A_{r-p'} = 0, \qquad (p' < p). \tag{13}$$

Then by taking successively

$$r = n + p' - p, \quad n + p' - p + 1, \cdots, n$$

in this equation and subtracting the results multiplied into appropriate constants from (5), the last $p - p' + 1$ terms of (5) can be removed, and a relation is thereby obtained which connects only p' coefficients of S. This relation may then be combined with (13) so as still further to reduce the number of coefficients linearly connected, and so on. It will be observed that the process is exactly parallel to that by which the highest common factor of the polynomials $B_p(x)$ and $B_{p'}(x) = 1 - b'_1 x - \cdots - b'_{p'} x^{p'}$ is derived. Since also the coefficients of x^i and of A_{n-i} at the outset of the two processes differ only by infinitesimals, it is clear that the same holds true after the first subtractions. It follows then by mathematical induction that they will differ also by infinitesimals at the close of the parallel processes. *If therefore there coexist two linear relations between the coefficients of S whose limiting forms correspond to two polynomials $B_p(x)$ and $B_{p'}(x)$, there must also be a linear relation whose limiting form corresponds to the highest common factor of these polynomials.*

There can not, however, be two linear relations which correspond to polynomials without a common polynomial factor. For the last subtraction in the process of finding their highest common factor would give a constant as a remainder. Consequently, to the last minuend and subtrahend there would correspond two linear relations whose limiting forms would differ only in a single coefficient, which is clearly impossible. By combining the two results last italicized we conclude finally that *there is a single characteristic linear relation between the coefficients of S which contains a smaller number of coefficients than any similar expression.*

5. *Fundamental expression for the function defined by the series.*

We now return to the question whether the series S can be rearranged so as to take the form (8). A sufficient condition for such a rearrangement is contained in the following theorem of WEIERSTRASS:* Given a series:

$$F = \sum_{q=0}^{\infty} u_q,$$

of which the general term u_q is a power series:

$$u_q = \sum_{n=0}^{\infty} a_{qn} x^n;$$

if the separate series u_q and the collective series F converge within a circle (R) and if also the latter series converges uniformly along every concentric circle

* BIERMANN, *Theorie der analytischen Functionen*, § 30, p. 146, or HARKNESS AND MORLEY, *Analytic Functions*, § 81.

(R_1) of radius $R_1 < R$, then within the circle (R) the collective series can be expressed as a power series:

$$F = \sum_{n=0}^{\infty} A_n x^n,$$

in which the general coefficient A_n is the sum of the corresponding coefficients a_{qn} of the component series u_q.

In the collective series which we condensed in (8) u_{q+1} is the power series for $\epsilon_{q+m} x^{q+m}/B_\mu(x)$, which converges within the circle C. Within this circle the collective series $F(x)$ evidently converges in the same manner as $\sum_{n=m}^{\infty} \epsilon_n x^n$. But the latter, being a power series, converges uniformly within any circle concentric with its circle of convergence and having a smaller radius. We infer therefore the equivalence of our two expressions under the following condition:

If $\sum_{n=m}^{\infty} \epsilon_n x^n$ converges within a circle whose radius is equal to or less than the radius of C, the series S will within this circle be identical with $F(x)$.

The inequalities (11) show that $\sum_{n=m}^{\infty} \epsilon_n x^n$ and S have a like connection with (12), and the former series, as well as S, must accordingly converge within the circle C. Hence *upon the hypothesis of a limiting form for our linear relation the function $F(x)$ and the series S are identical within the circle C.*

6. *The radius of convergence.*

One important inference can now be drawn from the form of (8). If $B_\mu(x)$ *is taken to denote the polynomial which corresponds to the linear relation containing the minimum number of coefficients of S, the roots of $B_\mu(x)$ will be, in general, singular points of the function defined by S. Hence the distance ρ from the origin to the nearest root (or roots) of $B_\mu(x) = 0$ is in general the radius of convergence.*

Attention should, however, be called to certain *exceptional cases*, in which the radius of convergence is greater than this distance. This may occur if one or more of the roots of $B_\mu(x)$ are zeros of the analytic function defined by $G(x) + \sum_{n=m}^{\infty} \epsilon_n x^n$. Such a root, at least if simple, is then no longer a singularity. That such exceptional cases do arise may be seen by multiplying together

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

and

$$\frac{1}{x-1} = \frac{1}{x} + \frac{1}{x^2} + \cdots.$$

The resulting LAURENT'S series is one in which for all values of n, positive or negative,

$$A_{n+1} = \left(1 - \frac{1}{n+1}\right) A_n + \frac{A_{n-1}}{n+1}.$$

Confining our attention to the part of the series which consists of the positive powers, we have a series which converges over the entire plane. Yet it is characterized by a limiting linear relation which corresponds to the polynomial $1 - x$.

7. *Series defining functions whose only singularities upon the circle of convergence are poles.*

From inspection of (8) it is evident that all the singularities of the function defined by our series S which lie upon its circle of convergence will be poles when $\sum_{n=m}^{\infty} \epsilon_n x^n$ has a radius of convergence greater than that of S. We shall show, conversely, that *if a series defines a function whose only singularities upon the circle of convergence are poles, the series is characterized by a limiting linear relation,* $\sum_{n=m}^{\infty} \epsilon_n x^n$ *having a radius of convergence greater than the distance from the origin to the nearest root of the polynomial which corresponds to the linear relation.*

To demonstrate the existence of a limiting linear relation, let the function be expressed as the sum of a rational fraction whose poles lie upon the circle of convergence and of a series $C_0 + C_1 x + C_2 x^2 + \cdots$ which converges beyond this circle. We will denote the denominator of the rational fraction by $B_p(x)$. Decompose the rational fractions into simple fractions $D_i(1 + xa_i^{-1})^{-r_i}$ and expand each simple fraction into a series in ascending powers of x. The $(n+1)$th coefficient will be

$$D_{in} = (-1)^n \frac{r_i(r_i+1)\cdots(r_i+n-1)D_i}{n!\, a_i^n}. \tag{14}$$

If now $r_i > r_j$, we obtain from (14)

$$\frac{|D_{in}|}{|D_{jn}|} \leqq \frac{|D_i|}{|D_j|} \cdot \frac{r_i + n - 1}{r_j}.$$

When, therefore, n is indefinitely increased, D_{jn} becomes negligible in comparison with D_{in}. We will accordingly resolve our rational fraction into two parts, the first of which is the sum of all simple fractions in which r_i has the maximum value, while the other part comprises the remaining fractions.

Let the expansions in series for the two parts be $\sum C'_n x^n$ and $\sum C''_n x^n$. If the former part consists of k simple fractions, we have

$$C'_n = D_{1n} + D_{2n} + \cdots + D_{kn}, \tag{15}$$

a like equation holding for C''_n. As n increases C''_n becomes infinitesimal in comparison with each of the k components of C'_n. The same is true of $C''_{n-i} (i = 1, 2, \cdots, k)$, inasmuch as the ratio between D_{jn} and $D_{j,\,n-i}$, any two

corresponding components of C''_n and C''_{n-i}, approaches a finite limit. We will show also that the same quantities become infinitesimal not simply in comparison with the components of the corresponding coefficients C'_{n-i} but also in comparison with at least one of these coefficients.

If possible, suppose the contrary to hold, and consider the system of simultaneous equations obtained by substituting $n-1$, $n-2$, $\cdots$, $n-k$ for n in (15). When n is indefinitely increased, the left hand member of each equation should become an infinitesimal part of any term on the other side of the equation. Introducing into the system the values of the $D_{i,n}$ furnished by (14) we have

$$\delta_i = a_1^i(a_1^{-n}D_1) + a_2^i(a_2^{-n}D_2) + \cdots + a_k^i(a_k^{-n}D_k), \qquad (i=1, 2, \cdots k),$$

in which the δ_i designate the negligible left hand members. Now the determinant of the coefficients of the quantities $a_i^{-n}D_i$ becomes, after the removal of the factor $a_1a_2\cdots a_k$, a well known expression for the product of the differences of the a_i. Since this product does not vanish, the system of equations can be satisfied only by supposing that each of the quantities $a_i^{-n}D_i$ is an infinitesimal part of itself—that is, only if $D_1 = D_2 = \cdots = D_k = 0$. Our initial hypothesis was therefore untenable. Some one (or more) of the coefficients $C'_{n-i}(i=1, 2, \cdots, k)$ must remain of the same order of magnitude as $D_{1,n-i}$, and in comparison with it the k quantities C''_{n-i} may be neglected.

A glance at (14) shows that this same coefficient is at least as great as $\gamma|a_1|^{-n}$, in which γ designates an appropriate finite number independent of n. Since the radius of convergence of $C_0 + C_1x + C_2x^2 + \cdots$ is greater than $|a_1|$, we have also $|C_n| < \gamma'|a'|^{-n}$, in which $|a'| > |a_1|$. It follows that the k coefficients C_{n-i} are likewise infinitesimal in comparison with the above coefficient.

The existence of a limiting linear relation between the coefficients of the series representing our function can now be quickly established. For if $B_{p'}(x)$ denotes the product of the denominators of the k simple fractions whose exponents had the greatest value, we have

$$C'_n = b'_1C'_{n-1} + b'_2C'_{n-2} + \cdots + b'_{p'}C'_{n-p'}.$$

By virtue of this relation the general coefficient $C_n + C'_n + C''_n$ in the series for the function—call it A_n—can be thrown into the form:

$$A_n = b'_1A_{n-1} + b'_2A_{n-2} + \cdots + b'_{p'}A_{n-p'} - (b'_1C''_{n-1} + b'_2C''_{n-2} + \cdots + b'_{p'}C''_{n-p'}) - (b'_1C_{n-1} + b'_2C_{n-2} + \cdots + b'_{p'}C_{n-p'}) + C''_n + C_n.$$

But, by the preceding paragraphs, when n increases, C_n, C''_n, and the two parentheses must become an infinitesimal part of some of the preceding terms. There is therefore a linear relation which corresponds to the polynomial $B_{p'}(x)$. The same must also be true of $B_p(x)$ since it contains $B_{p'}(x)$ as a factor.

It has thus been shown that the existence of a limiting linear relation for which $\sum_{m=n}^{\infty} \epsilon_n x^n$ has a greater radius of convergence than S is a necessary as well as a sufficient condition that all the singularities on the circle of convergence should be poles. The positions of the poles and their orders are determined by the polynomial $B_p(x)$ which corresponds to the lowest linear relation of this character. It is, however, to be emphasized that *unless all the poles are of the first order, this will not be the limiting linear relation containing the smallest number of terms. The latter is dependent only upon the poles of highest order.*

8. *Hadamard's conditions for polar singularities upon the circle of convergence.*

The necessary and sufficient conditions that there should be p poles upon the circle of convergence of a given series $A_0 + A_1 x + A_2 x^2 + \cdots$, and no other singularities, have been previously studied* by HADAMARD and were thrown by him into the following very interesting form:

If ρ is the radius of convergence of the series and

$$\Delta_{n,l} = \begin{vmatrix} A_n & A_{n+1} & \cdots & A_{n+l} \\ A_{n+1} & A_{n+2} & \cdots & A_{n+l+1} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n+l} & A_{n+l+1} & \cdots & A_{n+2l} \end{vmatrix},$$

then the greatest affix of a point of condensation of the sequence

$$|\Delta_{n,l}|^{1/n} \qquad (n = 1, 2, \cdots)$$

must be equal to

$$\text{I)} \quad 1/\rho^{l+1} \quad (l < p),$$

$$\text{II)} \quad 1/\rho^{p}\rho' \quad (l = p),$$

in which ρ' is a number greater than ρ.

The second of these conditions can easily be shown to be a necessary one with the aid of our linear relation. For this purpose subtract from the last column of $\Delta_{n,p}$ the preceding column multiplied into b_1, the second preceding column multiplied into b_2 and so on. The constituents of the last column then become the quantities ϵ_{n+p+i} $(i = 0, 1, \cdots, p)$. They must therefore be less than $\gamma'/(\rho')^{n+p+i}$ while each constituent of any preceding column is less than γ/ρ^i. The modulus of $\Delta_{n,p}$ must accordingly be less than $\Gamma/\rho^{np}(\rho')^p$, in which Γ denotes an appropriate number independent of n. The desired conclusion now immediately follows. Moreover $|\Delta_{n,l}|$ is less than $\Gamma/\rho^{(n+l)}$ for any value of l.

* HADAMARD, Liouville's Journal, ser. 4, vol. 8, p. 119.

The greatest affix of a point of condensation of the above sequence can therefore never exceed $1/\rho^{l+1}$.

To demonstrate that the conditions I and II together are sufficient, HADAMARD proves successively the following three points:

1) When n increases $|\Delta_{n,p-1}|^{1/n}$ converges toward ρ^{-p}.

2) Simultaneously the solutions of the system of equations

$$A_{n+p+i} + x_1^{(n)} A_{n+p+i-1} + \cdots + x_p^{(n)} A_{n+i} = 0 \qquad (i = 0, 1 \cdots, p-1) \tag{16}$$

approach definite limits,* these limits being the negatives of the coefficients b_i of $B_p(x)$.

3) $$\epsilon_{n+p} \equiv A_{n+p} - b_1 A_{n+p-1} - \cdots - b_n A_n < \left(\frac{1+\epsilon}{\rho'}\right)^n,$$

in which ϵ decreases indefinitely as n increases.

The last point shows that the singularities can be removed by multiplying the series with $B_p(x)$, and thus that they are poles.

HADAMARD has no occasion to explicitly note the existence of a limiting linear relation of the character here considered. This is, however, implied in 2).

As regards the significance of the above three points, the first is used by HADAMARD as a lemma by which to establish the second under the given conditions. Logically, however, the second always carries with it the first, though the converse does not seem to be true. If, in fact, $|x_p^{(n)}|$ or $|\Delta_{n+1,p-1}|/|\Delta_{n,p-1}|$ approaches the limit $1/\rho^p$, we have

$$\frac{(1-\epsilon)^j}{\rho^{pj}} < \frac{|\Delta_{n+j,p-1}|}{|\Delta_{n,p-1}|} < \frac{(1+\epsilon)^j}{\rho^{pj}}.$$

Hence the limit of $|\Delta_{n+j,p-1}|^{1/n+j}$ for $j=\infty$ is $1/\rho^p$.—The third point carries with it the two preceding.

It should be expressly noted that *no one of the three points is necessary that there should be a limiting relation.* The second, however, suffices, and this fact suggests one method of determining the existence of a linear relation for a given series, namely, by an examination of the determinant ratios which are the solution of (16) to see whether for any value of p they converge toward definite limits.

9. *The linear relation with an infinite number of terms.*

The theory developed in the preceding sections can be extended in a restricted form to series whose coefficients satisfy a linear relation which tends to take the form:

$$A_n = b_1 A_{n-1} + b_2 A_{n-2} + b_3 A_{n-3} + \cdots, \tag{17}$$

* HADAMARD's notation has been varied to bring it into accord with that used here.

the number of terms in the relation increasing indefinitely with n. In place of $B_p(x)$ an infinite series $B(x) = 1 - b_1x - b_2x^2 \cdots$ must then be introduced. Let, for convenience, the linear relation be written thus:

$$A_n = (b_1A_{n-1} + b_2A_{n-2} + \cdots + b_nA_0) + \epsilon_n,$$

in which

$$\epsilon_n = \epsilon_1^{(n)}c_1A_{n-1} + \epsilon_2^{(n)}c_2A_{n-2} + \cdots + \epsilon_n^{(n)}c_nA_0.$$

Suppose now that the quantities $\epsilon_i^{(n)}$ and c_i satisfy the following conditions:

(1) For any arbitrarily assigned positive quantity ϵ an integer m can be found such that for $n \geqq m$ every $|\epsilon_i^{(n)}|$ is less than ϵ.

(2) The series $\sum_{i=1}^{\infty} c_ix^i$ converges at least within the circle of convergence of

$$P(x) \equiv \frac{1}{B(x)}.$$

Should every ϵ_n $(n \geqq m)$ be neglected, the coefficients after the mth term would be formed in accordance with the law (17), and the series would be the expansion of some fraction $G(x)/B(x)$, in which $G(x)$ denotes a polynomial. By collecting, as in § 2, the parts of the successive terms of the series which explicitly contain ϵ_n $(n = m, m+1, \cdots)$, we obtain the function,

$$\frac{G(x)}{B(x)} + \frac{\sum_{n=m}^{\infty} \epsilon_n x^n}{B(x)}$$

in place of (8). Let γ and ξ have meanings similar to those in which they were before employed. Then

$$|A_0| < \gamma, \quad |A_1| < \gamma\xi^{-1}, \quad \cdots, \quad |A_{m-1}| < \gamma\xi^{-(m-1)},$$

and

$$|\epsilon_m| < \epsilon\gamma\left(\frac{|c_1|}{\xi^{m-1}} + \frac{|c_2|}{\xi^{m-2}} + \cdots + \frac{|c_{m-1}|}{\xi} + |c_m|\right) = \frac{\epsilon\gamma X'}{\xi^m},$$

where

$$X' = |c_1|\,\xi + |c_2|\,\xi^2 + \cdots + |c_m|\,\xi^m.$$

As ξ lies within the circle of convergence of $\sum_{i=1}^{\infty} c_ix^i$, the sum of the series:

$$|c_1|\,\xi + |c_2|\,\xi^2 + |c_3|\,\xi^3 + \cdots$$

must have a finite value, say V. Then if $X = \epsilon\gamma V$, we have

$$|\epsilon_m| < X\xi^{-m}.$$

With this value of X the argument of § 3 may now be repeated, and our series may thus be proved to converge within a circle having the origin as its center and a radius equal to the distance from the origin to the nearest singular point of $1/B(x)$.

In case condition (2) is not fulfilled, ξ must be taken smaller than the radius of convergence of $\sum_{i=1}^{\infty} c_i x^i$ in order that the above considerations shall apply, and the series $A_0 + A_1 x + A_2 x^2 + \cdots$ is shown merely to converge within this circle. No difficulty will be experienced in extending the same reasoning also to relations of the form:

$$A_n = \left(b_1 + \sum_{i=1}^{i=k} \epsilon_{1i} c_{1i}\right) A_{n-1} + \left(b_2 + \sum_{i=1}^{i=k} \epsilon_{2i} c_{2i}\right) A_{n-2} + \cdots, \tag{18}$$

the quantity V being replaced by $\sum_{i=1}^{i=k} V_i$. The result which will thus be reached may be summed up as follows:

II. *Let a series $S = A_0 + A_1 x + A_2 x^2 + \cdots$ be given such that, after some fixed term, each coefficient is connected with the preceding coefficients by a linear relation* (18) *in which $b_1, b_2, \cdots$ are the coefficients of a convergent series $\sum_{n=1}^{\infty} b_n x^n$. If*

(1) *for any arbitrarily assigned positive quantity ϵ an integer m can be found such that every $|\epsilon_{ni}|$ for $n \geqq m$ will be less than ϵ, and*

(2) *the k series $\sum_{n=1}^{\infty} c_{ni} x^n$, $(i = 1, 2, \cdots, k)$, converge within a circle whose radius is the distance from the origin to the nearest singular point of*

$$\frac{1}{1 - b_1 x - b_2 x^2 - \cdots},$$

then the series S converges within this circle. In case only condition (1) *is fulfilled S converges within the region of convergence common to the k series.*

10. *Series in two or more variables.*

Criteria similar to I and II can be derived for the convergence of power series containing two or more variables. This will be shown here only for the case of two variables, inasmuch as the theory is in every respect parallel for the case of more than two variables. Let

$$B_p(x, y) = 1 - \sum b_{ij} x^i y^j, \qquad (0 < i + j \leqq p).$$

If the reciprocal of this polynomial be expanded into a series

$$1 + A_{10} x + A_{01} y + A_{20} x^2 + A_{11} xy + A_{02} y^2 + \cdots, \tag{19}$$

the coefficients of the terms of any given degree are obtained from the coefficients of the terms of lower degree by means of the recurring relation:

$$A_{ij} = \sum b_{qr} A_{i-q,\, j-r}, \qquad (0 < q + r \leqq p).^*$$

If in any series such a relation holds after some fixed term, the series is the expansion of some rational function which has $B_\mu(x, y)$ for its denominator.

Consider now a series S of the form (19) in which every coefficient A_{ij} whose index sum $i + j$ is equal to or exceeds some fixed integer M is derived from the coefficients having a smaller index sum by a relation of the form:

$$A_{ij} = \sum b_{qr} A_{i-q,\, j-r} + \epsilon_{ij}, \qquad (0 < q + r \leqq p),$$

where

$$\epsilon_{ij} = \sum \epsilon_{qr}^{(ij)} A_{i-q,\, j-r}. \tag{20}$$

Let $G(x, y)/B_\mu(x, y)$ represent the rational fraction obtained by omitting every ϵ_{ij} for which $i + j$ exceeds some fixed integer M. Then by reasoning similar to that of § 2 it can be shown that the coefficient of any term $x^i y^j$ in S can be obtained by summing the corresponding coefficients in the various power series which represent the separate members of the function:

$$F(x, y) = \frac{G(x, y)}{B_\mu(x, y)} + \sum_i \sum_j \frac{\epsilon_{ij} x^i y^j}{B_\mu(x, y)}, \qquad (i + j \geqq M). \tag{21}$$

Denote with R and R' the radii of any two circles in the x- and y-planes respectively which together constitute a domain of convergence for the series expanding $G(x, y)/B_\mu(x, y)$ and $1/B_\mu(x, y)$. Then if ξ and η are any positive quantities smaller than R and R' respectively, the coefficient of $x^i y^j$ in either series will be less than $\gamma/\xi^i \eta^j$, where γ is some fixed number independent of i and j. We have then

$$|A_{ij}| < \frac{\gamma}{\xi^i \eta^j}, \qquad (i + j < M).$$

Suppose now that for any arbitrarily small ϵ a value can be found for M such that for $i + j \geqq M$, every $|\epsilon_{qr}^{(ij)}|$ will be less than ϵ. It will follow then from (20) that for every combination of values for i and j whose sum is M we have

$$|\epsilon_{ij}| < \frac{X}{\xi^i \eta^j}$$

where

$$X = \epsilon\gamma(\xi + \eta + \xi^2 + \xi\eta + \eta^2 + \cdots + \eta^r),$$

and therefore also

$$|A_{ij}| < \frac{\gamma(1 + X)}{\xi^i \eta^j}.$$

Taking for i and j all combinations of values whose sum successively is M, $M + 1$, $M + 2$, $\cdots$, we obtain finally

* No negative subscript is to be admitted either here or elsewhere.

$$|\epsilon_{ij}| < \frac{X(1+X)^{i+j-M}}{\xi^i \eta^j}, \qquad (i+j \geqq M)$$

$$|A_{ij}| < \frac{\gamma(1+X)^{i+j-M+1}}{\xi^i \eta^j}.$$

The last inequalities show that S converges with

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(1+X)^{i+j} x^i y^j}{\xi^i \eta^j}.$$

Now this double series is the expansion for

$$\frac{1}{\left(1-\frac{x(1+X)}{\xi}\right)\left(1-\frac{y(1+X)}{\eta}\right)}$$

and converges for

$$|x| < \frac{\xi}{1+X}, \qquad |y| < \frac{\eta}{1+X}.$$

But ϵ and with it X can be made less than any arbitrarily small positive quantity by taking M sufficiently large. We have therefore the following result:

III. *Let a series*

$$S = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_{ij} x^i y^j$$

be given in which every coefficient A_{ij} whose index sum $i+j$ exceeds a fixed number is obtained from the coefficients of smaller index sum by a relation of the form

$$A_{ij} = \sum_{q,\,r} (b_{qr} + \epsilon_{qr}^{(ij)})\, A_{i-q,\,j-r}, \qquad (0 < q+r \leqq p).$$

If for any assigned positive value ϵ, however small, an integer M can be found such that for $i+j \geqq M$ every $|\epsilon_{pq}^{(ij)}|$ is less than ϵ, the series will converge at every point within the domain of convergence of $P(x, y) \equiv 1/B_\mu(x, y)$.

Upon the same hypothesis S can be proved to be identical with the function $F(x, y)$. The proof is based upon a generalization * of the theorem of WEIERSTRASS cited in § 5. The form of (21) shows that if $B_\mu(x, y)$ *is irreducible, the above mentioned domain is, in general, the domain of convergence of* S, although we have not excluded the possibility of exceptional cases in which there may be a greater domain. If $B_\mu(x)$ is reducible, a linear relation of lower order may correspond to one of its factors, from which a greater domain of convergence might be argued.

The domain of convergence of the series for $1/B_\mu(x, y)$ is fixed by the theorem of WEIERSTRASS concerning the reciprocal of a power series. We have

* BIERMANN, loc. cit., pp. 148-150.

only to seek for values (x', y') which satisfy the equation $B_\rho(x, y) = 0$ and make $|B_\rho(x, y)| > 0$ for $|x| < |x'|$ and $|y| < |y'|$. For definiteness and brevity the term *radius of convergence* * is sometimes applied to a power series in two variables, being used to designate the maximum quantity ρ such that the series converges if $|x| < \rho$ and $|y| < \rho$. Its value is, in general, determined by $B_\rho(x, y)$. Thus for the simple case in which $B_\rho(x, y) = 1 - b_1x - b_2y$, the radius of convergence is obviously $1/(|b_1| + |b_2|)$.

Theorem III permits of extension in the same manner as I, giving the following result:

IV. *Let a series*

$$S = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} A_{ij} x^i y^j$$

be given in which every coefficient having an index sum greater than some fixed number is obtained from the coefficients having a smaller index sum by a relation of the form:

$$A_{ij} = \sum_q \sum_r b_{i-q,\, j-r} A_{i-q,\, j-r} + \epsilon_{qr}^{(ij)} (c_1 m_{ij}^{(1)} + \cdots + c_\rho m_{ij}^{(\rho)}) A_{i-q,\, j-r}.$$

If for an arbitrarily assigned positive quantity ϵ *an integer* M *can be found such that for* $i + j \geqq M$ *every* $|\epsilon_{qr}^{(ij)}|$ *is less than* ϵ, *the series will converge within the region of convergence common to the* p *series* $\sum_i \sum_j m_{ij}^{(k)} x^i y^j$, $(k = 1, 2, \cdots, p)$ *and to the series for the reciprocal of*

$$1 - \sum \sum b_{ij} x^i y^j, \qquad (i + j > 0).$$

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* BIERMANN, loc. cit., p. 138.

ON THE EXISTENCE OF THE GREEN'S FUNCTION FOR THE MOST GENERAL SIMPLY CONNECTED PLANE REGION

BY

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The problem of mapping the interior of a simply connected plane region T conformally on the interior of a circle depends for its solution on the proof of the existence of the Green's function corresponding to T, i. e., of a function u satisfying the following conditions:

(1) Except at a single arbitrarily chosen interior point O of T, u is single valued and harmonic throughout T. By *harmonic* is meant that u satisfies Laplace's equation, $\Delta u = 0$;

(2) At the point O, u becomes discontinuous like $\log 1/r$, or

$$u = \log 1/r + u',$$

where u' is harmonic throughout the neighborhood of O, r denoting the distance of the variable point (x, y) from O;

(3) The boundary values of u are all 0, i. e., when the point (x, y) approaches from the interior of T an arbitrary point of the boundary of T, u approaches the value 0.

Hitherto the existence of the Green's function has been established (a) for regions T bounded by a finite number of pieces of analytic curves (SCHWARZ), and (b) for regions T bounded by a finite number of pieces of regular curves (PARAF and PAINLEVÉ). By a *regular* curve is meant one that has a tangent that turns continuously as a variable point traces out the curve. Cf. PICARD, *Traité*, vol. II, chs. III, IV, X. In these cases it is shown furthermore that the boundary of T is transformed continuously into the boundary of the circle. In the present paper it is proposed to establish the existence of the Green's function for the most general† simply connected plane region T. Such regions include (α) regions whose boundary is any Jordan curve;‡ (β) regions whose boundary is more

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† A single exceptional case is to be noted. If the image of T on the sphere of stereographic projection consisted of the complete sphere less *one* point, the theorem would not hold.

‡ By a *Jordan curve* is meant a curve of the general class of continuous curves without multiple points, considered by JORDAN, *Cours d'Analyse*, vol. I, 2d edition, 1893, p. 90. Cf. HURWITZ's geometric statement of JORDAN'S results, Zurich address, International Congress of Mathematicians, 1897.

general. As an example of the latter class of regions consider the following. Assume on the axis of x a set of points perfect, but not dense throughout any interval whatsoever.* Erect at each point of this set a perpendicular of unit length, drawn in the direction of the positive axis of y. The part of the upper half plane that remains after these lines have been cut out forms a region T in point. If, in particular, the set of points on the axis of x was so chosen that its content is positive, the content of the boundary of T, regarded as a set of points lying in a two-dimensional space, will likewise be positive. Moreover, the points of this boundary present a peculiarity not shared by the points of a Jordan curve. There are namely points in this set that must be regarded as points of the boundary of T, in as much as portions of T lie in every neighborhood of these points; and yet it is impossible to approach these points from the interior of T along a continuous curve; that is to say, if A denotes such a point and a circle of suitable radius ϵ is described with A as center, then it is, of course, possible to join an arbitrary point, say O, of T with a point of T lying in the circle by means of a curve lying wholly within T. But if now a second circle of suitable radius ϵ' is drawn about A, it will not be possible to continue the above path to a point in the inner circle without first passing out of the larger circle. In the above example, any interior point on a line about which lines cluster on both sides will serve as a point A.

The existence of points A makes it necessary to define more precisely what form condition (3) in the definition of the Green's function shall take on. I will therefore state this condition as follows: If p_1, p_2, $\cdots$ are any set of points lying within T and having as their only limiting point (Häufungsstelle) the point A; if, moreover, the value of u in p_n be denoted by u_{p_n}, then

$$\lim_{n=\infty} u_{p_n} = 0.$$

Proof of the Existence Theorem.†—We will first assume for the sake of simplicity that T lies wholly in the finite region of the plane. Let the plane be divided into equal squares, 2^{-n} on a side, by means of the axes of x and y and lines drawn parallel to these axes. When n is taken sufficiently large, some of these squares will lie wholly within T. Let C be such a square and denote by C_n the largest simply connected region that can be formed out of C and other

* A suitable set for present purposes is one of which I have made frequent use; cf. American Journal of Mathematics, vol. 19, 1897, p. 167; *Cambridge Colloquium*, Bulletin of the American Mathematical Society, ser. 2, vol. 5, 1898: Lecture VI, p. 82.

† The method used in this proof is in substance the same as the one given by POINCARÉ in his paper on the representation of an analytic configuration: $w = f(z)$, by means of a pair of single valued functions of a parameter: $z = \phi(t)$, $w = \psi(t)$; Bulletin de la Société mathématique de France, vol. 11, 1883, p. 112. Cf. *Cambridge Colloquium*, above cited, Lectures II, III.

squares lying, like C, within T. When n increases indefinitely, any preassigned point of T finally will be included in C_n.

Denote by u_n the Green's function for C_n having O as pole (n being taken large enough so that O will be an interior point of C_n). Then u_n is positive throughout C_n and increases at each point of C_n when n increases. In fact, consider the function

$$u_{n+1} - u_n$$

throughout C_n. This function is harmonic at every interior point of C_n and approaches continuous boundary values along the boundary of C_n that are sometimes positive and never negative. Hence the function is positive throughout the whole interior of C_n. Furthermore, u_n converges toward a limit, as will presently be shown, when n becomes infinite, and thus by HARNACK's theorem* the limit u of u_n:

$$\lim_{n=\infty} u_n = u,$$

is a function that satisfies at least the first two conditions that the Green's function for T must fulfill. We shall show that u also satisfies the third condition and thus is the desired function. First, however, to show that u_n actually approaches a limit. The proof for the special case we are considering (namely, where T is of finite extent) can be given at once by considering, say, the Green's function U for a circle with O as center and large enough to contain the whole region T. For the difference

$$U - u_n$$

is positive throughout the whole extent of C_n. Thus u_n, always increasing with n, never exceeds U and hence approaches a limit at a single point, say, of C, so that HARNACK's theorem is now applicable.

This mode of proof is typical for the general case, but it is necessary to use a more complicated function U to establish the convergence. Such a function we will now introduce, choosing it at once in such a way that it can be used to show that the limiting function u satisfies condition (3) too.

Let a, b, c be any three points of the plane of the complex variable z, and let the plane be cut from a to c along the arc abc of the circle passing through these three points. Then the plane, thus bounded, can be mapped conformally on a quadrilateral of a second plane (the w-plane), this quadrilateral being formed by a triangle whose sides are arcs of circles tangent to each other at the

* Harnack's theorem asserts that if a function u_n is harmonic throughout a region T for all values of $n \geqq m$ and if u_n increases at all points of the region when n increases; if furthermore at one single interior point of T u_n approaches a limit when n becomes infinite, then u_n converges at all interior points of T and the limit u is harmonic throughout the interior of T. Cf. HARNACK, *Logarithmisches Potential*, Leipzig, 1887, p. 167.

vertices, and an adjacent triangle formed by reflecting the first triangle in one of its sides. Construct now by successive reflections the complete Riemann's surfaces that belong to the analytic function and its inverse, defined by the conformal transformation just considered. The result is the familiar modular figure* of the w-plane with the infinitely many-leaved surface of the z-plane having logarithmic branch points at a, b, c.

Now choose at pleasure a point O in an arbitrary leaf of the z-surface distinct from the points a, b, c, and denote its image in the w-plane by O'. Let U' be the Green's function for the fundamental circle of the w-figure, having its pole in O', and denote the same function transferred to the z-surface, by U. U is then a function single-valued and harmonic on the z-surface except at the one point O, where it becomes logarithmically infinite; and, moreover, *U approaches the value* 0 *when the point* (x, y) *approaches any one of the points* a, b, c *along any path whatever, or, more generally, when* (x, y) *passes through the points of a set* $p_1, p_2, \cdots$, *lying at pleasure on the surface, but having as the only point about which they cluster one of the points* a, b, c.

Returning now to the region T, let us choose three points in the boundary of T at pleasure and denote them by a, b, c. Form the function U just defined for these three points and for the point O already selected as the pole O of U. Then U, regarded as a function restricted to the domain T, is *single valued* throughout this domain, becomes logarithmically infinite in O, and is positive along the whole boundary of any C_n. Hence

$$u_n < U$$

and thus u_n approaches a limit $u \leqq U$ when n becomes infinite.

Finally, it can be shown by means of the function U that u approaches the boundary value 0 when the point (x, y) approaches a, or more generally, when a set of points $p_1, p_2, \cdots$ is so taken in T that $\lim p_n = a$, when $n = \infty$. For the corresponding values of U: $U_{p_1}, U_{p_2}, \cdots$ approach 0 as their limit, and $0 < u \leqq U$.

Thus the proof of the existence theorem is complete in case T lies wholly in the finite region of the plane, or what is in substance the same thing, in case there exist points of the plane external to T and its boundary. If, however, T and its boundary cover the complete sphere of stereographic projection, the proof requires a slight modification in the construction of the regions C_n. It will no longer be feasible to associate with C *all* the squares that lie in T and form with C a simply connected region, for this region might extend to infinity. It is sufficient, however, to confine oneself to those squares that lie wholly in T and simultaneously in a circle with center O and radius n.

* For an elementary discussion of the transformation and the functions thus defined cf. *Cambridge Colloquium*, above cited, Lecture I.

The proof is now complete. It can readily be shown that the function u thus determined is the only function satisfying the three conditions for the Green's function. Thus the most general transformation that carries the *interior* of a circle over conformally into itself, no assumption being made about the *boundary*, is a linear transformation; and hence the conformal transformation of the interior of an arbitrary simply connected region T on the interior of a circle depends on three real parameters. These may be taken as the coördinates of the point O_1 of the circle into which O is transformed and the angle that the image of a specified curve through O makes with the radius of the circle drawn to O_1. It remains to investigate whether the boundary of T (or a piece of it) is a Jordan curve, this boundary goes over continuously into the boundary of the circle.

The theorem of this paper justifies the conclusion that the function $\phi(t)$ employed by POINCARÉ* may be taken as an automorphic function and makes probable that the analytic configuration $w = f(z)$ may still be represented in a satisfactory manner by means of the parameter t in the neighborhood of its singular points. Such a neighborhood will not, to be sure, be represented by the neighborhood of an interior point of the t-circle. But if it turns out, as I hope to show later, that it is represented by the neighborhood of a parabolic vertex, a desirable result will have been established.

ON THE ATLANTIC,
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* loc. cit.

"D" LINES ON QUADRICS*

BY

ALEXANDER PELL

§ 1. *Introduction.*

§ 1. The "D" lines so named by E. COSSERAT † were considered originally by DARBOUX.‡ They are the lines drawn upon a surface in such a manner that the osculating sphere at every point is tangent to the surface at that point. From this definition DARBOUX deduced their differential equation:

$$\frac{3d^2s}{ds} = \frac{2\mathrm{S}\,d^2x\,d\frac{\partial F}{\partial x} + \mathrm{S}\,dx\,d^2\frac{\partial F}{\partial x}}{\mathrm{S}\,dx\,d\frac{\partial F}{\partial x}}. \tag{1}$$

The summations refer to the coördinates x, y, z; $F(x, y, z) = 0$ being the equation of the surface under consideration.

Applying this equation to the quadric surfaces we obtain

$$\frac{1}{D^2}\,ds^2 = \mathrm{S}\,dx\,d\frac{\partial F}{\partial x}. \tag{2}$$

In the same memoir DARBOUX showed that by using elliptic coördinates the variables in (2) can be separated. The equation then becomes

$$\frac{\sqrt{D^2 - u}\,du}{\sqrt{(a^2 - u)(b^2 - u)(c^2 - u)}} = \frac{\sqrt{D^2 - v}\,dv}{\sqrt{(a^2 - v)(b^2 - v)(c^2 - v)}}. \tag{3}$$

Here $u =$ constant, $v =$ constant are the equations of the lines of curvature of the quadric, and a, b, c are its semi-axes.

These lines have attracted considerable attention; and ENNEPER,§ RIBAUCOUR,‖ and E. COSSERAT ¶ have attempted to study them upon surfaces in general.

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† Comptes Rendus, 1895.

‡ Comptes Rendus, 1871.

§ Göttinger Nachrichten, 1871.

‖ Comptes Rendus, 1875.

¶ loc. cit.

The object of this paper is to apply the theory of elliptic functions to the integration of the differential equation (3) given by DARBOUX and to obtain an idea of the appearance of these lines and some of their properties.

§§ 2, 3. "*D*" *lines on central quadrics.*

§ 2. The geometrical interpretation of (2) for the case under consideration gives the following theorem:

I. *For any central quadric the semi-diameter, drawn parallel to a tangent to a "D" line on that surface, is constant in length.*

Conversely, let us find the differential equation of the curves on the surface of a central quadric such that the semi-diameter parallel to the tangent to these curves is constant in length.

We have for any central quadric,

$$\frac{1}{D^2} = \frac{\cos^2 \omega}{u} + \frac{\sin^2 \omega}{v}, \tag{4}$$

where u, v are the parameters of the lines of curvature, ω is the angle of the curve with the lines of curvature v = constant, and D the semi-diameter parallel to the tangent to the curve. We suppose D constant.

If E and G are the first fundamental coefficients of Gauss's theory and

$$f(u) = 4(a^2 - u)(b^2 - u)(c^2 - u),$$

then

$$E = \frac{u(u-v)}{f(u)}, \quad G = \frac{v(v-u)}{f(v)},$$

$$\tan \omega = \sqrt{-\frac{f(u)\, v\, dv}{f(v)\, u\, du}}.$$

Substituting into (4) we shall have

$$\frac{\sqrt{D^2 - u}\, du}{\sqrt{f(u)}} = \frac{\sqrt{D^2 - v}\, dv}{\sqrt{-f(v)}},$$

and since this equation coincides with (3) we can say:

II. *The "D" lines on central quadrics are the only lines satisfying the conditions of the theorem* I.

§ 3. The differential equation (3) when transformed into polar coördinates:

$$x = a \cos \phi \sin \psi,$$
$$y = b \sin \phi \sin \psi,$$
$$z = c \cos \psi,$$

becomes

$$d\psi^2 + \sin^2 \psi\, d\phi^2 = \frac{ds^2}{D^2}. \tag{5}$$

This equation leads easily to the theorem:

III. *The length of a portion of the central projection of the "D" line upon a concentric sphere of radius D is equal to the projected portion of the "D" line.*

§§ 4–11. *"D" lines on an ellipsoid.*

§ 4. The equation (3) we write as follows:

$$\frac{(D^2-u)du}{2\sqrt{R(u)}}+\frac{(D^2-v)dv}{2\sqrt{R(v)}}=0, \tag{6}$$

where

$$R(\theta)=(a^2-\theta)(b^2-\theta)(c^2-\theta)(D^2-\theta).$$

We suppose $a>b>c$. To (6) we join the following two:

$$\begin{cases}\dfrac{du}{2\sqrt{R(u)}}+\dfrac{dv}{2\sqrt{R(v)}}=dU,\\[2ex] \dfrac{ds}{dU}=-D\sqrt{(u-D^2)(v-D^2)},\end{cases} \tag{7}$$

where U is a new parameter, used by WEIERSTRASS; the second equation can be obtained from

$$ds=\sqrt{E}\,du\sin\omega+\sqrt{G}\,dv\cos\omega,$$

$$\sin\omega=\frac{\sqrt{(u-D^2)v}}{D\sqrt{u-v}},\qquad \cos\omega=\frac{\sqrt{(D^2-v)u}}{D\sqrt{u-v}}.$$

§ 5. Let $u=$ constant determine the confocal hyperboloid of two nappes, and $v=$ constant, that of one nappe. Then

$$a^2>u>b^2>v>c^2.$$

If $\sin\omega$ and $\cos\omega$ be real we must have

$$u>D^2>v.$$

Therefore three cases present themselves:

$$1)\ D^2=b^2,\quad 2)\ D^2>b^2,\quad 3)\ D^2<b^2.$$

The first case as indicated by DARBOUX gives us the circular sections of the ellipsoid. Theorem I gives the same result and leads to the following theorem:

IV. *Through every umbilic passes only one "D" line—the circular section parallel to the tangent plane at the opposite umbilic.*

§ 6. Passing over to the second case, in which $D^2>b^2$, we see that in (6) and (7) u varies from D^2 to a^2, v from c^2 to b^2.

To accomplish the inversion of the system (6) and (7) we use the substitution given by HALPHEN* for the case when the roots of the quartic are known. Put

$$\theta - a^2 = -\frac{\wp'(\frac{1}{2}\lambda)}{\wp(t) - \wp(\frac{1}{2}\lambda)}, \quad \theta - D^2 = -\frac{\wp'(\frac{1}{2}\lambda)}{\wp(\frac{1}{2}\lambda) - e_3}\,\frac{\wp(t) - e_3}{\wp(t) - \wp(\frac{1}{2}\lambda)},$$

$$\theta - b^2 = -\frac{\wp'(\frac{1}{2}\lambda)}{\wp(\frac{1}{2}\lambda) - e_2}\,\frac{\wp(t) - e_2}{\wp(t) - \wp(\frac{1}{2}\lambda)},$$

$$\theta - c^2 = -\frac{\wp'(\frac{1}{2}\lambda)}{\wp(\frac{1}{2}\lambda) - e_1}\,\frac{\wp(t) - e_1}{\wp(t) - \wp(\frac{1}{2}\lambda)},$$

When θ varies from D^2 to a^2, we must take t purely imaginary and let it vary from ω_3 to 0; when θ varies from c^2 to b^2, $t - \omega_1$ must be purely imaginary and must vary from 0 to ω_3; ω_1 and ω_3 are the half-periods of the elliptic functions introduced, the first real, the second purely imaginary.

Let t_1 be the parameter t corresponding to u; t_2 that corresponding to v. Then t_1 varies from ω_3 to 0; t_2 from ω_1 to $\omega_3 + \omega_1$.

We can easily write:

$$(8)\qquad \begin{cases} e_1 = \frac{1}{12}\{(D^2 - a^2)(c^2 - b^2) + (b^2 - a^2)(c^2 - D^2)\} > 0, \\ e_2 = \frac{1}{12}\{(D^2 - a^2)(b^2 - c^2) + (c^2 - a^2)(b^2 - D^2)\} > 0, \\ e_3 = \frac{1}{12}\{(b^2 - a^2)(D^2 - c^2) + (c^2 - a^2)(D^2 - b^2)\} < 0. \end{cases}$$

Also

$$\frac{d\theta}{dt} = \frac{\wp'(\frac{1}{2}\lambda)\wp'(t)}{[\wp(t) - \wp(\frac{1}{2}\lambda)]^2}, \qquad \pm\sqrt{R(\theta)} = i\frac{d\theta}{dt}.$$

Therefore

$$\frac{du}{\sqrt{R(u)}} = idt_1, \qquad \frac{dv}{\sqrt{R(v)}} = -idt_2.$$

On substituting the above values into (6) and integrating we have

$$(9)\qquad e^{2v_1(t_1 - t_2)\left(\frac{\wp'(\frac{1}{2}\lambda)}{\wp(\frac{1}{2}\lambda) - e_3} - 2\zeta(\frac{1}{2}\lambda)\right)} = \frac{\sigma(t_1 - \frac{1}{2}\lambda)\sigma(t_2 + \frac{1}{2}\lambda)}{\sigma(t_1 + \frac{1}{2}\lambda)\sigma(t_2 - \frac{1}{2}\lambda)},$$

$$-2Ui = t_1 - t_2.$$

Introduce another parameter a, such that

$$2ia = t_1 + t_2;$$

then

$$(10)\qquad \begin{cases} t_1 = (a - U)i, \\ t_2 = (a + U)i. \end{cases}$$

* HALPHEN, *Fonctions Elliptiques*, vol. I, p. 131.

Put also

$$-\frac{\wp'(\frac{1}{2}\lambda)}{\wp(\frac{1}{2}\lambda)-e_3}-2\zeta(\tfrac{1}{2}\lambda)=-2\frac{\sigma_3'(\frac{1}{2}\lambda)}{\sigma_3(\frac{1}{2}\lambda)}=K.$$

Then (9) can be reduced to

$$\wp(ai)=\frac{\wp(Ui+\frac{1}{2}\lambda)\sigma^2(Ui+\frac{1}{2}\lambda)-e^{2e_1-2Ui K}\wp(Ui+\frac{1}{2}\lambda)\sigma^2(Ui-\frac{1}{2}\lambda)}{\sigma^2(Ui+\frac{1}{2}\lambda)-e^{2e_1-2Ui K}\sigma^2(Ui-\frac{1}{2}\lambda)}.$$

§7. We shall use this equation to calculate the coördinates x, y, z of the points of the "D" lines as uniform functions of the parameter U.

Without going into calculations which should not trouble anyone familiar with elliptic functions we merely write the final results:

$$(11)\quad \begin{cases} \dfrac{x}{a}=\pm\dfrac{1}{2\sigma_3(\frac{1}{2}\lambda)}\cdot\dfrac{1}{\sigma(2Ui)}\Big(\sigma(2Ui+\frac{1}{2}\lambda)\cdot e^{-2Ui\frac{\sigma_3'(\frac{1}{2}\lambda)}{\sigma_3(\frac{1}{2}\lambda)}-e_1} \\ \qquad\qquad +\sigma(2Ui-\frac{1}{2}\lambda)\cdot e^{2Ui\frac{\sigma_3'(\frac{1}{2}\lambda)}{\sigma_3(\frac{1}{2}\lambda)}-e_1}\Big), \\ \dfrac{y}{b}=\pm\dfrac{1}{2\sigma_3(\frac{1}{2}\lambda)\sqrt{e_2-e_1}}\cdot\dfrac{1}{\sigma(2Ui)}\Big(\sigma_2(2Ui+\frac{1}{2}\lambda)\cdot e^{-2Ui\frac{\sigma_3'(\frac{1}{2}\lambda)}{\sigma_3(\frac{1}{2}\lambda)}-e_1} \\ \qquad\qquad -\sigma_2(2Ui-\frac{1}{2}\lambda)\cdot e^{2Ui\frac{\sigma_3'(\frac{1}{2}\lambda)}{\sigma_3(\frac{1}{2}\lambda)}-e_1}\Big), \\ \dfrac{z}{c}=\pm\dfrac{1}{2\sigma_3(\frac{1}{2}\lambda)\sqrt{e_1-e_2}}\cdot\dfrac{1}{\sigma(2Ui)}\Big(\sigma_1(2Ui+\frac{1}{2}\lambda)\cdot e^{-2Ui\frac{\sigma_3'(\frac{1}{2}\lambda)}{\sigma_3(\frac{1}{2}\lambda)}-e_1} \\ \qquad\qquad -\sigma_1(2Ui-\frac{1}{2}\lambda)\cdot e^{2Ui\frac{\sigma_3'(\frac{1}{2}\lambda)}{\sigma_3(\frac{1}{2}\lambda)}-e_1}\Big). \end{cases}$$

From (7) we obtain:

$$(12)\quad \frac{1}{D}\cdot\frac{ds}{d(2Ui)}=\frac{1}{2\sigma_3(\frac{1}{2}\lambda)}\cdot\frac{1}{\sigma(2Ui)}\Big(\sigma_3(2Ui+\tfrac{1}{2}\lambda)\cdot e^{-2Ui\frac{\sigma_3'(\frac{1}{2}\lambda)}{\sigma_3(\frac{1}{2}\lambda)}-e_1} -\sigma_3(2Ui-\tfrac{1}{2}\lambda)\cdot e^{2Ui\frac{\sigma_3'(\frac{1}{2}\lambda)}{\sigma_3(\frac{1}{2}\lambda)}-e_1}\Big).$$

The above expressions contain two arbitrary constants λ and c_1.

§8. The derivative $ds\,d(2Ui)$ satisfies LAMÉ'S equation:

$$\frac{1}{\phi}\frac{d^2\phi}{d(2Ui)^2}=2\wp(2Ui)+\wp(\tfrac{1}{2}\lambda-\omega_3).$$

§ 9. We have seen that t_1 and $t_2 - \omega_1$ must be purely imaginary and vary between ω_3 and 0. Hence $t_2 - t_1 = 2Ui$ must be a complex number of the form:

$$\omega_1 + k\omega_3 + iX,$$

where k is any whole number and X varies from 0 to $\pm \omega_3/i$.

Substituting this value for $2Ui$ in the formulæ (11) and distinguishing the cases of k even and odd we have the two following sets of expressions, in which A and C represent $-\frac{1}{2}K$ and $ie_1'A$ respectively:

1) k even:

$$\frac{x}{a} = \pm \frac{1}{2\sigma_3(\frac{1}{2}\lambda)} \cdot \frac{e^{-A(X-C)i}\sigma_1(iX + \frac{1}{2}\lambda) + e^{A(X-C)i}\sigma_1(iX - \frac{1}{2}\lambda)}{\sigma_1(iX)},$$

$$\frac{y}{c} = \pm \frac{1}{2\sigma_3(\frac{1}{2}\lambda)} \cdot \frac{e^{-Ai(X-C)i}\sigma_3(iX + \frac{1}{2}\lambda) - e^{A(X-C)i}\sigma_3(iX - \frac{1}{2}\lambda)}{\sigma_1(iX)},$$

$$\frac{z}{c} = \pm \frac{\sqrt{e_1 - e_3}}{2\sigma_3(\frac{1}{2}\lambda)} \cdot \frac{e^{-Ai(X-C)}\sigma(iX + \frac{1}{2}\lambda) + e^{Ai(X-C)}\sigma(iX - \frac{1}{2}\lambda)}{\sigma_1(iX)},$$

$$\frac{1}{D} \cdot \frac{ds}{dX} = \frac{\sqrt{e_1 - e_3}}{2\sigma_3(\frac{1}{2}\lambda)} \cdot \frac{e^{-Ai(X-C)}\sigma_2(iX + \frac{1}{2}\lambda) - e^{Ai(X-C)}\sigma_2(iX - \frac{1}{2}\lambda)}{\sigma_1(iX)}.$$

2) k odd:

$$\frac{x}{a} = \pm \frac{1}{2\sigma_3(\frac{1}{2}\lambda)} \cdot \frac{e^{-Ai(X-C)}\sigma_2(iX + \frac{1}{2}\lambda) + e^{Ai(X-C)}\sigma_2(iX - \frac{1}{2}\lambda)}{\sigma_2(iX)},$$

$$\frac{y}{b} = \pm \frac{\sqrt{e_2 - e_3}}{2\sigma_3(\frac{1}{2}\lambda)} \cdot \frac{e^{-Ai(X-C)}\sigma(iX + \frac{1}{2}\lambda) + e^{Ai(X-C)}\sigma(iX - \frac{1}{2}\lambda)}{\sigma_2(iX)},$$

$$\frac{z}{c} = \pm \frac{i}{2\sigma_3(\frac{1}{2}\lambda)} \cdot \frac{e^{-Ai(X-C)}\sigma_3(iX + \frac{1}{2}\lambda) - e^{Ai(X-C)}\sigma_3(iX - \frac{1}{2}\lambda)}{\sigma_2(iX)},$$

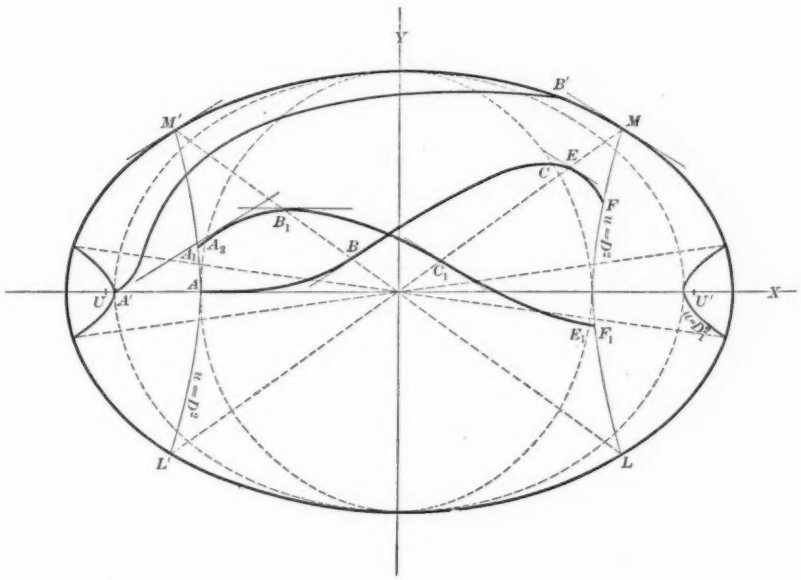
$$\frac{1}{D} \cdot \frac{ds}{dX} = \frac{\sqrt{e_2 - e_3}}{2\sigma_3(\frac{1}{2}\lambda)} \cdot \frac{e^{-Ai(X-C)}\sigma_1(iX + \frac{1}{2}\lambda) - e^{Ai(X-C)}\sigma_1(iX - \frac{1}{2}\lambda)}{\sigma_2(iX)}.$$

In these formulæ C and X are real, C varying from 0 to $\pm\infty$, and X from 0 to $\pm \omega_3 i$.

§ 10. It is easy to establish the following properties of these curves:

1) At the points of intersection of these curves with the lines of curvature $u = D^2$, they are orthogonal to these lines of curvature.

2) They do not extend beyond the two branches of the lines of curvature $u = D^2$.



3) The points corresponding to the final values 0 and ω_3/i of the parameter X lie upon two plane central sections tangent to the line $u = D^2$ at the points $u = D^2$, $v = c^2$ and $u = D^2$, $v = b^2$, respectively.

4) The tangents at these points are parallel to the elements of the cylinders touching the ellipsoid along the two plane sections.

5) Two curves corresponding to the values C and $C + (A\omega_3 - \frac{1}{2}\eta_3\lambda)/A$, have a common point and a common tangent on one of these plane sections.

6) Four such curves form a continuous "D" line between the two branches of $u = D^2$, having an inflectional tangent plane.

7) The curves $C = 0$ have a cuspidal point for $X = 0$.

8) Lines of curvature passing through the point of intersection of two lines corresponding to the same D bisect the angle between them.

In the figure ML, $M'L'$ are the projections on the XY-plane of the lines of curvature $u = D^2$; $ABCEF$ is the "D" line corresponding to $C = 0$; $M'L$, ML', AA_2CE_1 are plane sections; and $A'B'$ is a "D" line corresponding to a value of D nearly equal to b.

§ 11. The third case, in which $D^2 < b^2$, can be treated in exactly the same manner. The expressions for the coördinates will be

1) k even:

$$\frac{x}{a} = \pm \frac{1}{2\sigma_2(\frac{1}{2}\lambda)} \cdot \frac{e^{-A(X-C)i}\sigma(\frac{1}{2}\lambda + iX) + e^{A(X-C)i}\sigma(\frac{1}{2}\lambda - iX)}{\sigma_1(iX)},$$

$$\frac{y}{b} = \pm \frac{i}{2\sigma_2(\frac{1}{2}\lambda)} \cdot \frac{e^{-A(X-C)i}\sigma_2(\frac{1}{2}\lambda + iX) - e^{A(X-C)i}\sigma_2(\frac{1}{2}\lambda - iX)}{\sigma_1(iX)},$$

$$\frac{z}{c} = \pm \frac{\sqrt{e_1 - e_3}}{2\sigma_2(\frac{1}{2}\lambda)} \cdot \frac{e^{-A(X-C)i}\sigma_1(\frac{1}{2}\lambda + iX) + e^{A(X-C)i}\sigma_1(\frac{1}{2}\lambda - iX)}{\sigma_1(iX)}.$$

2) k odd:

$$\frac{x}{a} = \pm \frac{1}{2\sigma_2(\frac{1}{2}\lambda)} \cdot \frac{e^{-A(X-C)i}\sigma_2(\frac{1}{2}\lambda + iX) - e^{A(X-C)i}\sigma_2(\frac{1}{2}\lambda - iX)}{\sigma_3(iX)},$$

$$\frac{y}{b} = \pm \frac{\sqrt{e_2 - e_3}}{2\sigma_2(\frac{1}{2}\lambda)} \cdot \frac{e^{-A(X-C)i}\sigma(\frac{1}{2}\lambda + iX) + e^{A(X-C)i}\sigma(\frac{1}{2}\lambda - iX)}{\sigma_3(iX)},$$

$$\frac{z}{c} = \pm \frac{i}{2\sigma_2(\frac{1}{2}\lambda)} \cdot \frac{e^{-A(X-C)i}\sigma_3(\frac{1}{2}\lambda + iX) + e^{A(X-C)i}\sigma_3(\frac{1}{2}\lambda - iX)}{\sigma_3(iX)}.$$

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SUNDRY METRIC THEOREMS CONCERNING N LINES IN A PLANE*

BY

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The point of departure for this paper is furnished by the article of Professor F. MORLEY† in the April number of the Transactions. For the convenience of the reader, the notation of that memoir has been as much as possible followed; but it will be perceived that even where, in the opening sections of the present essay, the consequent resemblance to portions of the former rises to the point of identity of formulæ, the geometric meaning which underlies these is quite distinct, while in the later portions of the article it has been necessary to find forms of statement unlike those suited to the preceding work.

In the last section of the article quoted (p. 184) its author points out that the problems to which that memoir is mainly devoted arise from an initial combination of n lines by pairs, while a grouping by threes, fours, or higher numbers is possible. The present paper is concerned with the case in which the lines are originally grouped in threes, and has for its basic element, analogous to the intersection of two lines as treated in the former article, the center of a circle tangent to the three. I shall briefly indicate a new series of theorems which thus arises from an altered interpretation of formulæ practically identical with those of Professor MORLEY, including analogues to his own theorems upon center-circles, as well as to the chain of propositions associated with the names of STEINER, MIQUEL, KANTOR, and CLIFFORD; I shall then develop a relation by which the new theorems are connected with those of the foregoing case; and finally I shall devote some space to the simpler aspects of the added multiplicity of forms resulting from that character by which the case here treated is chiefly distinguished from the preceding, to wit, the assignment to each line of a definite direction, the reversal of which in any instance, while leaving the original configuration of n lines apparently unaffected, entirely changes the diagram of circles built upon it.

§1. *The system of n directed lines in a plane.*

The locus of the centers of circles tangent to two lines, when these are considered as each described indifferently in either of two opposite directions, con-

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† *On the Metric Geometry of the Plane n-Line*, Transactions of the American Mathematical Society, vol. 1, no. 2.

sists of the two axes of symmetry, or bisectors of the four angles at the intersection; but when we assign to each line a definite direction to the exclusion of the opposite, and require that a tangent circle shall be so situated that a point moving continuously around the latter shall, at the point of contact with a line, move in the direction in which that line is described, then the locus of the centers of such circles is reduced to a single axis. This is *equidistant* from the given lines, and will be distinguished by that property from its companion axis (to which alone, on the other hand, the name "bisector of the angle" is appropriate), if we consent to regard the distances of a point of the former from the two given lines as agreeing in sign, while those of a point of the latter are opposite to one another. Given three directed lines, the three axes, thus equidistant respectively from the lines of three pairs, meet in a point, the center of the one circle tangent to the lines. Though this may happen to be the circle inscribed in the triangle, or one of the escribed circles, its center is the sole point *equidistant* from the three directed lines. Given four directed lines, it is an elementary theorem—included, as will be seen, under a more general proposition presently to be stated—that the quadrangle formed by the four points each equidistant from three of the lines is inscriptible in a circle.

In expressing algebraically these elementary relations, I follow Professor MORLEY in using circular coördinates * x and y, defined by the equations

$$x = \mathrm{x} + i\mathrm{y},$$
$$y = \mathrm{x} - i\mathrm{y},$$

(where x and y are the ordinary rectangular coördinates) but it becomes convenient in this case to use as the standard equation of the right line a form different from that adapted to the requirements of his article. In place of

$$xt_1 + y = x_1 t_1$$

I employ

$$xt_1 - 2r_1 + \frac{y}{t_1} = 0;$$

so that, while the convention is retained that t_1 shall denote a complex quantity of absolute value 1, the t_1 of this article is t_1^2 in the notation of the other, while r_1 is a real quantity, equal to the length of the perpendicular from the origin to the line.

Then

$$\left(xt_1 - 2r_1 + \frac{y}{t_1}\right) - \left(xt_2 - 2r_2 + \frac{y}{t_2}\right) = 0$$

represents the axis equidistant from two given lines; and elimination between two such equations gives

*SALMON: *Higher Plane Curves*, p. 7.

$$x = \frac{2r_1t_1}{(t_1-t_2)(t_1-t_3)} + \frac{2r_2t_2}{(t_2-t_1)(t_2-t_3)} + \frac{2r_3t_3}{(t_3-t_1)(t_3-t_2)}$$

as the point equidistant from three lines, each determinate in direction.

We have here a sum of fractions closely resembling those formerly denoted * by the symbols a_1, etc. Accordingly, let us put

$$a_\alpha = \sum \frac{2r_1t_1^{n-\alpha-1}}{(t_1-t_2)\cdots(t_1-t_n)} \qquad (\alpha = 1, 2, \cdots, n),$$

noting, for later use, that the conjugate of a_α is

$$(-1)^{n-1}Ta_{n-\alpha},$$

where

$$T = t_1t_2\cdots t_n,$$

and that a_α for $n = m$ is given in terms of the a's for $n = m+1$ and t_{m+1} by the formula:

$$a_\alpha - a_{\alpha+1}t_{m+1}.$$

Then the position of the equidistant center of three lines is given by

$$x = a_1 \qquad (n = 3),$$

while the equation:

$$x = a_1 - a_2t \qquad (n = 4),$$

represents the circumcircle of the inscriptible quadrangle defined by four given directed lines. Its center and radius are

$$a_1 \text{ and } |a_2| \qquad (n = 4).$$

It is immediately evident, on writing out the formulæ, that the algebraic argument is here simply a repetition of that used by Professor MORLEY in the article quoted. I therefore omit it altogether, merely stating the theorems into which the equations are translated under the modified code of interpretation.

I. *Let there be given five right lines, each of specified direction. Any four of these determine a quadrangle (viz., the vertices of the latter are points in which the equidistant axes of pairs of lines, taken from the given four, converge by threes) which quadrangle is inscriptible in a circle C_4, and the five circumcircles C_4 have their centers on one new circle C_5. Six given directed lines, taken by fives, lead to six such circles C_5, whose centers lie on a new circle C_6; and so on indefinitely.*

II. *Given five directed right lines, the five circles C_4 of Theorem* I *meet in a point N_5; and in general, given n directed right lines, the n circles C_{n-1} meet in a point N_n.*

* loc. cit., pp. 99, 100.

III. *Given six directed right lines, the six points N_5 of Theorem* II *lie on a circle O_6; given seven such lines, the seven circles O_6 meet in a point P_7; given eight such lines, the eight points P_7 lie on a circle O_8; and so on in an infinite alternation of points and circles.*

The point P_{2p+1} is given by

$$\begin{vmatrix} a_1 - x & a_2 & \cdots & a_p \\ a_2 & a_3 & \cdots & a_{p+1} \\ \vdots & \vdots & & \vdots \\ a_p & a_{p+1} & \cdots & a_{2p-1} \end{vmatrix} = 0 ,$$

and the equation of the circle O_{2p+2} is

$$\begin{vmatrix} a_1 - x & a_2 & \cdots & a_p \\ a_2 & a_3 & \cdots & a_{p+1} \\ \vdots & \vdots & & \vdots \\ a_p & a_{p+1} & \cdots & a_{2p-1} \end{vmatrix} = t \begin{vmatrix} a_2 & a_3 & \cdots & a_{p+1} \\ a_3 & a_4 & \cdots & a_{p+2} \\ \vdots & \vdots & & \vdots \\ a_{p+1} & a_{p+2} & \cdots & a_{2p} \end{vmatrix} ,$$

but this circle will in the general case be replaced by a right line on condition that

$$\begin{vmatrix} a_3 & a_4 & \cdots & a_{p+1} \\ a_4 & a_5 & \cdots & a_{p+2} \\ \vdots & \vdots & & \vdots \\ a_{p+1} & a_{p+2} & \cdots & a_{2p-1} \end{vmatrix} = 0 .$$

The algebra leading to these theorems may be found in the article of Professor MORLEY in the following sections: for Theorem I, § 2; for II, § 3; for III, § 4.

[The condition that O_6 be a right line, viz.,

$$a_3 = 0 ,$$

is readily developed into the geometric statement that the six given lines must be tangents to a certain curve, parallel to the hypocycloid of fourth class (see its equation in SALMON, *Higher Plane Curves*, § 117, ex. 2), and it is thus shown that what might be inferred from the concluding section of Professor MORLEY's article to be a *sufficient* condition for such a breaking-up is also a *necessary* one. When the given lines are tangents to such a curve, the equations leading to the foregoing theorems assume a high degree of simplicity, and several subordinate theorems are inferred without difficulty; the results, however, do not seem of sufficient interest to warrant an extended discussion.]

It will be observed that the number of lines, points, or circles named in any one of the foregoing theorems is greater by one than in the corresponding theorem of the preceding article. When we take account of the lines, etc., that are involved in the steps anterior to that which the theorem directly states, these numbers are still further increased.

For instance, that part of Theorem III which forms an analogue to the theorem of MIQUEL would, if stated independently, read as follows:

Given six directed lines, they form fifteen pairs each having an equidistant axis; these axes converge by threes at twenty points, the vertices of fifteen inscriptible quadrangles, whose circumcircles meet by fives in six points, and these points lie on a circle O_6.

§ 2. *Inter-relation between the two series of theorems.*

It is shown in § 5 of the paper above cited that the MIQUEL diagram may be constructed in reverse order, beginning with the circle, on which five points are selected arbitrarily, and ending with the five lines assumed in the usual form of the theorem. This property also has its extension to the present case. To show this, it is necessary to find a set of six lines to which those deduced as above shall be equidistant axes; or, algebraically, to write a self-conjugate equation symmetrical in all but one of six t's, and having a turn for the coefficient of x, in such a form that on the subtraction of another equation, like it save for the particular t in which it is asymmetric, the result shall be the equation (13) of the memoir cited, viz.:

$$x + ys_4 = as_1 + \beta s_2 + as_3 .$$

The terms in x and y of such an equation may be constructed from the description, and it is then found possible to complete the equation. To represent the result, let A stand for the function

$$\frac{i}{t_6}\left(\frac{t_2t_3t_4t_5}{t_1} + \cdots + \frac{t_1t_2t_3t_4}{t_5}\right),$$

A' for the conjugate of the same, and B for the function produced when in the symmetrical expression

$$\tfrac{1}{2}\left(\frac{t_1t_2t_3}{t_4t_5t_6} + \frac{t_1t_2t_4}{t_3t_5t_6} + \cdots + \frac{t_4t_5t_6}{t_1t_2t_3}\right),$$

every term is multiplied either by i or by $1/i$ according as $1/t_6$ or t_6 is already a factor of the term. The desired equation will then be

$$\frac{t_1t_2t_3t_4t_5i}{t_6}x + \frac{t_6}{t_1t_2t_3t_4t_5i}y - Aa - B\beta - A'a = 0 .$$

From this subtract a like equation in which t_5 is treated as is t_6 in this; the result divided by $(t_5^2 - t_6^2)i/(t_5 t_6)$ is a form in which it is necessary to replace $1/t_1^2$, $1/t_2^2$, etc., by t_1, t_2, etc., and then the equation (13) above mentioned is obtained. Hence the theorem:

IV. *If a circle on which six points have been selected is made, by the construction quoted, the Miquel circle of six sets of lines, corresponding to the six ways of taking the given points by fives, the fifteen lines thus obtained will be equidistant axes to a new set of six lines.*

This theorem points directly to the inference that when we begin with six given directed lines, it is possible in six different ways to select five from among the fifteen equidistant axes belonging to the given lines in pairs, so that the Miquel circle of these five axes shall be identical with the circle O of the six given lines.

An examination proves that this inference is true, and that the five axes to be selected in any case are those which arise from pairing a given line with each of the others in succession. For let us assume one given line L and n others in a plane with it, these n being, like L, each determinate in direction; there are then n axes, each equidistant from L and one of the given lines. First, let $n = 3$; it is then known from Theorem I, or it can be proved in a variety of ways, that the circumcircle of the three axes contains also the point equidistant from the three lines. Now add a fourth line, and in consequence a fourth equidistant axis; there are four such circumcircles, meeting (as is known) in a point, and also (as shown by STEINER) having their centers on a circle. These four are circumcircles, not only of triangles formed by the axes, but of quadrangles formed by the given lines in conjunction with L. There is a fifth circumcircle, belonging to the quadrangle derived from the four lines without L, but this fifth circle would have been grouped with three of the others had a different line been chosen to play the part of L; and, since three are sufficient to determine both the common point of their circumferences and the circle through their centers, it is seen that the theorems I, II concerning the circumcircles of quadrangles are a necessary consequence of the known theorems on the circumcircles of triangles. When n is increased to 5 and upward, the same considerations apply; whence we are entitled to state the following theorem:

V. *If n lines, each fixed in direction, be given, and their equidistant axes be grouped in sets, those forming one set which are equidistant from a selected line and the other $n-1$ lines severally (so that, each axis being counted in two sets, the $\frac{1}{2}n(n-1)$ axes form n sets of $n-1$ axes apiece), then the centercircles of the n sets coincide in one circle C_n, the nodes of the n sets in one point N_n, and (if n be odd) their Clifford points in one point P_n or (if n be even) their Clifford circles in one circle O_n.*

Although this theorem seems to merge the subject-matter of the present

article in that of the preceding, rendering a separate examination unnecessary, save as an amplification of the field already explored, this is true only as regards the class of properties thus far discussed. I turn, accordingly, to other aspects which require an independent treatment; namely, those depending upon the effect of reversing the direction in which a given line has been assumed to be described.

§ 3. *Reversal of the direction of the given lines.*

When the second line of the pair:

$$xt_1 - 2r_1 + \frac{y}{t_1} = 0\,, \qquad xt_2 - 2r_2 + \frac{y}{t_2} = 0$$

is reversed in direction, the corresponding parenthesis in the equation of their equidistant axis changes sign from $-$ to $+$, producing

$$\left(xt_1 - 2r_1 + \frac{y}{t_1}\right) + \left(xt_2 - 2r_2 + \frac{y}{t_2}\right) = 0\,;$$

whence the effect of the reversal will be expressed in the resulting formulæ by change of sign of r_2 and t_2. It will be convenient to assume in the case of n lines having one line reversed, that this line is the n-th; and the modified values of the constants, obtained by changing r_n and t_n to $-r_n$ and $-t_n$, will be denoted by the mark $\hat{}$, thus, $\hat{a}_\alpha$. Then the relations of these constants to those appertaining to the original directions of the lines is given by

$$a_\alpha - a_{\alpha+1}t_n = \hat{a}_\alpha + \hat{a}_{\alpha+1}t_n\,;$$

each member of the equation representing the value of a_α for the $n-1$ unreversed lines.

Two circles C_n (see Theorem I) derived on the respective suppositions that the n-th line is not reversed, and that it is reversed, have the equations:

$$x = a_1 - a_2 t\,, \qquad x = \hat{a}_1 + \hat{a}_2 t\,.$$

These coincide at the point which, while the number of lines was only $n-1$, was designated as a_1, now called indifferently

$$a_1 - a_2 t_n \qquad \text{or} \qquad \hat{a}_1 + \hat{a}_2 t_n\,.$$

Their angle at this intersection is that of the strokes extending from their respective centers; hence it is the amplitude of

$$\frac{-a_2 t_n}{\hat{a}_2 t_n} = -\frac{a_2}{\hat{a}_2}\,.$$

To obtain this amplitude separate from the ratio of absolute values, we divide this expression by its conjugate, and take the square root of the result. Now

the conjugate of a_2 is $(-1)^{n-1}Ta_{n-2}$, but that of $\hat{a}_2$ is $(-1)^{n-2}T\hat{a}_{n-2}$; whence that of $-a_2\hat{a}_2$ is $a_{n-2}/\hat{a}_{n-2}$; and the angle of the circles is the amplitude of

$$\sqrt{\frac{-a_2\hat{a}_{n-2}}{a_{n-2}\hat{a}_2}}.$$

This affords no simple result except in the case $n=4$, when the amplitude derived is that of i, viz., $\frac{1}{2}\pi$; and we learn that *two such circles cut orthogonally.* Now the circle C_4 is not only the first member of the series of circles C_n, but stands in the same relation to the circles O_n, viz., O_6, O_8, etc., and the inquiry is suggested whether the property of orthogonal section under the given conditions, though not extending beyond the first term of the former series, may not prove to belong to subsequent terms of the series O. This is found to be the fact, as follows:

The equation of O_{2p+2} is:

$$x=\frac{D_2}{D_1}-\frac{D_3}{D_1}t,$$

where D_1, D_2, D_3 are the determinants:

$$D_1=\begin{vmatrix} a_3 & \cdots & a_{p+1} \\ \vdots & & \vdots \\ a_{p+1} & \cdots & a_{2p-1} \end{vmatrix}, \qquad D_2=\begin{vmatrix} a_1 & \cdots & a_p \\ \vdots & & \vdots \\ a_p & \cdots & a_{2p-1} \end{vmatrix}, \qquad D_3=\begin{vmatrix} a_2 & \cdots & a_{p+1} \\ \vdots & & \vdots \\ a_{p+1} & \cdots & a_{2p} \end{vmatrix}.$$

When one line has been reversed, thus changing the signs of r_{2p+2} and t_{2p+2}, the equation of the new circle will be a similar formula which we may write

$$x=\frac{\Delta_2}{\Delta_1}+\frac{\Delta_3}{\Delta_1}t.$$

As in the case of the circles C_n, the two circles O have a common point (here one of the points P_{n-1}) which is fixed by the $n-1$ unreversed lines. The quantity whose amplitude furnishes the angle of section is the product of

$$\frac{-D_3\Delta_1}{D_1\Delta_3}$$

by the ratio of the values of t corresponding, in the two circles severally, to this common point. These values in the former case were alike (viz., t_4); here, though not equal to t_{2p+2}, they are again alike, being the product of t_{2p+2} by quantities dependent on the first $2p+1$ lines only.

To show this it is necessary to rehearse briefly some of the formulæ of § 4 of Professor MORLEY'S paper, and in quoting them a slight modification of their notation will adapt them to the present applications without interfering with reference to the original form. Let s_1, s_2, $\cdots$, s_m represent the sum, sum of

products by twos, etc., of m quantities, τ', τ'', etc., upon which no restriction is at first placed save that the absolute value of each is unity. Then the equation

$$x = a_1 - a_2 t \qquad (n = 4),$$

here representing a circle C_4, is one of five included in

$$x = a_1 - a_2 s_1 + a_3 s_2 \qquad \binom{n=5}{m=2},$$

in the sense that when one of the two τ's is identified with the constant t_4, the other with the variable t, the latter equation reduces to the preceding. A common point N_5 of these circles is obtained by showing that one τ may still be variable, though x be made constant, if the other τ satisfy two equations into which the one last written is in that case resolved, viz.:

$$\begin{aligned} x &= a_1 - a_2 \tau' \\ 0 &= a_2 - a_3 \tau' \end{aligned} \qquad (n = 5).$$

These are in the same way included in

$$\begin{aligned} x &= a_1 - a_2 s_1 + a_3 s_2 \\ 0 &= a_2 - a_3 s_1 + a_4 s_2 \end{aligned} \qquad \binom{n=6}{m=2}.$$

For here, if τ'' be t_6, the equations become identical with the preceding pair, the latter of which shows the value which is thus imposed on τ', viz., a_2/a_3, $(n = 5)$. The equation of O_6, viz.,

$$\begin{vmatrix} x - a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = t \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} \qquad (n = 6)$$

is a direct consequence of the last pair of equations, t being put for $s_2 = \tau' \cdot \tau''$ and therefore consisting, for the point at which this pair falls back into the preceding, of the product of t_6 by a_2/a_3 $(n = 5)$.

This process is repeated indefinitely. The equations

$$\begin{aligned} x &= a_1 - a_2 s_1 + a_3 s_2 - a_4 s_3 \\ 0 &= a_2 - a_3 s_1 + a_4 s_2 - a_5 s_3 \end{aligned} \qquad \binom{n=7}{m=3},$$

into which the preceding pair is expanded, will again admit of the location of P_7, by variable τ together with a constant x, if they still hold true when broken up into the following three:

$$\begin{aligned} x &= a_1 - a_2 s_1 + a_3 s_2 \\ 0 &= a_2 - a_3 s_1 + a_4 s_2 \\ 0 &= a_3 - a_4 s_1 + a_5 s_2 \end{aligned} \qquad \binom{n=7}{m=2}$$

The second and third of these fix upon s_2, or $\tau'\tau''$, the value

$$\frac{\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}}{\begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix}} \qquad (n=7);$$

and thus determine that in the equation of O_8, where the variable t is $\tau'\tau''\tau'''$, the value for this variable which will designate the particular point P_7 must be the product of $\tau''' = t_8$ by that value of $\tau'\tau''$ which has just been written, and in which the constants, being all taken for $n = 7$, are independent of t_8. And so in general.

Returning now to the question of the angle at this common point, which was to be found as the amplitude of a quotient, we see that the factor derived from the t's is unity, whence the amplitude depends wholly on the remaining factor already given, viz.:

$$\frac{-D_3\Delta_1}{D_1\Delta_3}.$$

We proceed, as before, to divide by the conjugate. The conjugates of D_1, D_3, Δ_1, Δ_3 are respectively

$$[(-1)^{2p+1}T]^{p-1}D_1, \quad [(-1)^{2p+1}T]^pD_3, \quad [(-1)^{2p}T]^{p-1}\Delta_1, \quad [(-1)^{2p}T]^p\Delta_3.$$

The net result of the division is -1; whence the derived angle is the amplitude of $\surd -1$, viz., $\frac{1}{2}\pi$. We have accordingly the following theorem, which forms the basis of all that is herein obtained concerning the reversal of lines.

VI. *If one of $2p+2$ given lines be described successively in opposite directions, the directions of the other $2p+1$ remaining fixed, the two circles O_{2p+2} determined by them will cut each other orthogonally. As respects this property, the circle C_4 (where $p=1$), is to be regarded as belonging to the series O.*

A circle O_{2p+2} determined by $2p+2$ directed lines has $2p+2$ circles orthogonal to it, each obtained by reversing one given line: whence $(p+1)(2p+1)$ circles will be obtained by reversing two lines, etc. The sum of these numbers (the coefficients of the $2p+2$ power of a binomial) is 2^{2p+2}, but because the result of reversing a given number of the lines is the same as that of reversing the remaining lines without the former, this sum is to be halved; whence the effect of regarding all the lines as subject to reversal is to replace the single circle O_{2p+2} by 2^{2p+1} circles. These form two sets of 2^{2p} circles each, characterized by the property that all circles orthogonal to any circle of one set are found in the other set, because any given arrangements of directions can be reached from any other only by an odd or else only by an even number of reversals. For example, when six lines are given, and each is considered as reversible in direction, they form fifteen pairs, each having two equidistant axes; the meeting of each axis

with any one of those sixteen others which are derived from line-pairs having one line in common with its own pair would give $(30 \times 16)\,2 = 240$ points if all were distinct, but as they coincide in threes, actually eighty points, equidistant from three lines apiece. Each of these points is associated with the thirty-six others having two lines in common with it in $(80 \times 36)/(3\,! \times 4) = 120$ sets of four, lying upon as many circles C_4. The circles C_n meet by fives in points N_5; and of these, on a like principle of enumeration, there are ninety-six. Finally, those points N_5 lie on thirty-two circles O_6, which form two sets of sixteen, each circle of one set finding all of the six circles orthogonal to it in the set opposite to its own.

§ 4. *Four lines, each susceptible of reversal.*

The simplest case to which the foregoing principles can be applied—that, namely in which $p = 1$, or n, the number of given lines, is four—presents some features peculiar to itself. Here a circle C_4 is drawn through the four points which are equidistant from the given lines taken in threes. There are eight such circles when the lines are reversed; while the number of equidistant points is sixteen, since each belongs to two circles. The circles form two sets of four, every member of either set being orthogonal to every member of the other. It follows immediately that each set is a coaxial system, the pair of real points in which one set meets being inverse points in respect to any circle of the other set. An advantage is obtained if the four lines be regarded as tangents of one parabola, which evidently imposes no restriction upon their generality, but enables us to employ axes of coördinates in a determinate relation to the given lines.

The equation of a tangent to a parabola is very simply obtained from the property that the foot of a perpendicular from the focus lies on the tangent at the vertex. The focus being the origin and the vertex $x = \frac{1}{4}$, this equation is

$$t_1 x - \frac{t_1}{1 + t_1^2} + \frac{y}{t_1} = 0\,.$$

The intersections of two such tangents is

$$x = \frac{1}{(1 + t_1^2)(1 + t_2^2)}\,,$$

their equidistant axis is

$$t_1 t_2 x + y = \frac{t_1 t_2 (1 - t_1 t_2)}{(1 + t_1^2)(1 + t_2^2)}\,,$$

and the point equidistant from three of them is

$$x = \frac{1 - t_2 t_3 - t_3 t_1 - t_1 t_2}{(1 + t_1^2)(1 + t_2^2)(1 + t_3^2)}\,.$$

From the last expression is derived immediately the equation of C_4, which may be written,

$$x = \frac{1}{P}[1 - \sigma_2 + \sigma_4 - (\sigma_1 - \sigma_3)t],$$

where

$$\sigma_1 = t_1 + t_2 + t_3 + t_4, \quad \sigma_2 = \Sigma t_1 t_2, \text{ etc.},$$

and

$$P = (1 + t_1^2)(1 + t_2^2)(1 + t_3^2)(1 + t_4^2).$$

If we consider the expressions for any two opposite vertices of the quadrilateral formed by the given lines, for instance,

$$x = \frac{1}{(1 + t_1^2)(1 + t_2^2)}, \qquad x' = \frac{1}{(1 + t_3^2)(1 + t_4^2)},$$

we observe that they satisfy the equation :

$$xx' = \frac{1}{P}.$$

This equation defines a complex involution in the plane, three pairs of which are composed of the vertices just named. Another pair is formed by the points

$$x = \frac{1 - t_2t_3 - t_3t_1 - t_1t_2}{(1 + t_1^2)(1 + t_2^2)(1 + t_3^2)}, \qquad x' = \frac{1}{(1 + t_4^2)(1 - t_2t_3 - t_3t_1 - t_1t_3)}.$$

Of this pair, the former, as has just been noted, is one of the points through which the circle C_4 is passed; but it may now be shown that the latter also lies on that circle. By writing x' in place of x and solving the equation of C_4 for t, we obtain an expression, which will or will not be a value actually assumed by t, according as its absolute value is or is not equal to unity. In writing this expression it is convenient to put $s_1 = t_1 + t_2 + t_3$, $s_2 = t_2t_3 + t_3t_1 + t_1t_2$, $s_3 = t_1t_2t_3$, and to retain σ_1, etc., in the sense already stated as expressions involving four variables. We then have

$$\frac{s_1\sigma_1 - s_1\sigma_3 - s_3\sigma_1 + s_3\sigma_3}{\sigma_1 - s_2\sigma_1 - \sigma_3 + s_2\sigma_3} = 1$$

as the condition that the circle C_4 contains the specified point. To verify this equation it is only necessary to replace s_1, s_2, etc., σ_1, σ_2, etc., by their conjugates, s_2/s_3, s_1/s_3, etc., σ_3/σ_4, σ_2/σ_4, etc., and it will be found that the fraction is changed into its reciprocal, which proves the proposition. It has been noted that two of the eight circles C_4 contain the point x; hence both contain x' as well, so that we can in future refer to the latter point as the *second intersection* of these circles. On the other hand, there are four points x on a circle C_4, and to define the partner circle of C_4 in the involution it is necessary to

use no more than three points x'; hence it is apparent that C_4 is its own partner, or a double circle of the involution.

But a double circle either contains the two double points of the involution or else they are inverse in respect to it; whence we come again upon the arrangement of the eight circles C_4 in two coaxial systems, and learn beside that the expression for the two points in which the four circles of one system meet is

$$x = \frac{1}{\sqrt{P}} = \frac{1}{\sqrt{(1+t_1^2)(1+t_2^2)(1+t_3^2)(1+t_4^2)}}.$$

The center of the involution, midway between these, is the origin, i. e., the focus of the tangent parabola, as is directly evident from the equation:

$$xx' = \frac{1}{P}.$$

By the aid of this equation all lines and circles of the plane are mapped into their partner circles. The twelve equidistant axes meeting by threes in points where the circles C_4 meet by pairs, are represented by as many circles, passing by threes through the second intersections of the same pairs of circles, passing also through the origin, and passing moreover by orthogonal pairs through vertices of the given quadrilateral opposite to those in which their partner axes cross at right angles.

The foregoing results relating to the general quadrilateral may be summed up as follows:

VII. *Given any four lines, there pass through their six intersections twelve lines (axes) each equidistant from two of the original four, and these twelve converge by threes in sixteen points. Through these points pass eight circles, four of which form a coaxial system through two real points, while the other four form a second coaxial system orthogonal to the first. The intersection of the two lines of centers is the focus of the parabola touching the four given lines. The sixteen points above mentioned lie by fours on these circles, each point having two circles through it, and the remaining intersections of the same circles constitute a second set of sixteen points. These lie by fours on twelve other circles, each point having three of these circles through it, while the twelve circles all pass through the focus of the parabola and meet orthogonally in pairs at the six intersections of the given lines.*

The case of four given lines is also susceptible of a different method of treatment leading to another configuration of circles. The proof is embraced in the general discussion in the next section; but the result for four lines only is considered worth stating separately, on account of its simplicity.

VIII. *If to three of any four given lines perpendiculars be drawn at the points where they respectively meet the fourth, the circumcircles of the two homolo-*

*gous triangles thus formed cut orthogonally, and the center of homology is one of the two points of section. The other point of section remains unchanged when a different line of the given four is used as an axis of homology, and is therefore common to all the circles of four orthogonal pairs.**

§ 5. *Any even number of lines, each reversible.*

The theorem VIII just stated for four lines may be generalized, and found to hold good for any higher even number. For this purpose the following lemma will be found serviceable.

LEMMA. *If a rectangle $ABCD$ have a vertex A at a fixed point, which is a $(p-1)$-fold point of a curve Q of p-th order, another vertex B upon the curve Q, a third C upon a fixed right line GH, which cuts Q at a point G and is at right angles with AG, then the locus of the fourth vertex D is a new curve of p-th order, having a $(p-1)$-fold point at A and an ordinary point at G.*

While circular coördinates might be employed in the proof of this proposition, they present no advantage over the ordinary Cartesian coördinates, and the demonstration is accordingly indicated in terms of the latter, designated as x and y. If A be taken as origin, and AG as axis of abscissas, the equation of Q may be written

$$a_0x^p + pa_1x^{p-1}y + \tfrac{1}{2}p(p-1)a_2x^{p-2}y^2 + \cdots + pa_{p-1}xy^{p-1} + a_py^p$$
$$- [b_0x^{p-1} + (p-1)b_1x^{p-2}y + \cdots + (p-1)b_{p-2}xy^{p-2} + b_{p-1}y^{p-1}] = 0.$$

The fixed line GH and the moving line AB have respectively the equations $x = b_0/a_0$ and $y = mx$.

The coördinates of B, the equation of BC, and coördinates of C are now readily formed, and the equation of CD written as parallel to AB. Then the locus of D follows by eliminating m between the last equation and that of AD, viz., $m = -x/y$.

It is

$$a_0(a_0y^p - pa_1xy^{p-1} + \cdots \pm a_px^p)$$
$$+ b_0[pa_1y^{p-1} - \tfrac{1}{2}p(p-1)a_2xy^{p-2} + \cdots \mp a_px^{p-1}]$$
$$- a_0y[(p-1)b_1y^{p-2} - \tfrac{1}{2}(p-1)(p-2)b_2xy^{p-3} + \cdots \pm b_{p-1}x^{p-2}] = 0.$$

The application to the present discussion comes through the theorem proved by CLIFFORD at the end of his memoir, *Synthetic Proof of Miquel's Theorem* (*Mathematical Papers*, p. 38), whence it appears that a curve such as Q is the

* It can be proved that the four points, which successively become centers of homology when the given lines in turn are used as axes, are also situated upon a ninth circle through the common point of the foregoing eight; but the demonstration of this, not being included in the general discussion of the following section, is omitted for the sake of brevity.

pedal of a p-fold parabola with regard to its focus: so that the lines BC and CD above are tangents to two p-fold parabolas having in common a fixed tangent GH, upon which the moving tangents meet at right angles; and it is seen that two such parabolic curves have a common focus A.

Now let there be given $2p + 2$ lines, from which one line L is arbitrarily selected and its direction assigned. At the point in which L is met by K, any one of the remaining $2p + 1$, let us suppose that an auxiliary directed line is drawn, so that L and this auxiliary shall have K as their equidistant axis, a construction which obviously demands only the doubling of a given angle. Then the circle O_{2p+2} of the set made up from L and the $2p + 1$ auxiliary lines is the same as the Clifford circle of the remaining given lines [Theorem V]. To each of the latter let a perpendicular be drawn at the point at which it meets L. These perpendiculars make up a new set of equidistant axes resulting from the former set by the reversal of the line L; consequently that circle O_{2p+2} which is identical with the Clifford circle of this set meets the former circle O_{2p+2} at right angles [Theorem VI]. The point P_{2p+1} belonging to the auxiliary lines taken by themselves is the same for one circle O as for the other, since these lines are all unreversed; consequently it forms one of the points of section of these two circles. Further, it is proved by CLIFFORD, in the memoir cited, that the Clifford circle of the $2p + 1$ given lines, L omitted, passes through the focus of the p-fold parabola which touches all the given lines. For the same reason, the Clifford circle of the $2p + 1$ perpendiculars passes through the focus of a second p-fold parabola, touching the perpendiculars and L. But these two p-fold parabolas, as we have inferred from the lemma, have a common focus. This focus, then, is the second point of section of the two circles. (It cannot be identical with the first, since its position depends upon L, while that of the former, P_{2p+1}, did not.) If now all the given lines be made to assume successively the part hitherto assigned to L, we have $2p + 2$ pairs of circles O intersecting orthogonally, and one of their points of section, the focus of the p-fold parabola touched by all, is the same for all the pairs.

The foregoing argument applies to all values of p, but it may be remarked that in the case $p = 1$, the curve Q of the lemma becomes the general right line, not passing through A at all; when $p = 2$, Q is a conic and A an ordinary point of it; for higher values A is a multiple point as described.

The result now obtained accordingly includes Theorem VIII as a special case, and (dropping from the statement the auxiliary lines) may be summarized as follows:

IX. *If all but one of an even number of given lines be turned, each through a right angle, about the point at which it meets the one remaining line, and if the Clifford circles be constructed for these rotated lines in both their original position and that to which they are turned, these circles intersect orthogonally.*

By making each line in succession the one upon which the others are turned, as many orthogonal pairs of circles are obtained as there are given lines, and all these circles pass through a common point, which is the focus of the p-fold parabola tangent to all the given lines.

HAVERFORD COLLEGE,
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AN APPLICATION OF GROUP THEORY TO HYDRODYNAMICS*

BY

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It has been observed by SOPHUS LIE that the stationary motion of a fluid can serve as a perfect picture of a one-parameter group in three variables. So far as I know, neither he nor any of his followers utilized this fact for the purposes of hydrodynamics. It is the purpose of the present paper to do this. One of the advantages gained for hydrodynamics by this standpoint lies in the general conception. But another advantage is, as is always the case when a class of problems is investigated from a new standpoint, that from the group-theoretical point of view, certain special cases are of exceptional interest, simplicity, and importance, cases which otherwise would appear difficult and unpromising.

§1. *Relation between the steady motion of a fluid and the theory of one-parameter groups.*

From the definition of the steady flow of a fluid, and LIE's conception of a one-parameter group, the following theorems will be seen to be true.

1. *Let x, y, z be the Cartesian coördinates of any point of the fluid at the time t, and let a, b, c be the coördinates of the same material point at the time $t = 0$. If*

$$(1) \qquad x = \phi(a, b, c; t), \quad y = \psi(a, b, c; t), \quad z = \chi(a, b, c; t)$$

are the equations which describe the steady motion of any point of the fluid, they are also the finite equations of a one-parameter group, the parameter being t.

2. *Conversely, if equations* (1) *are the equations of a one-parameter group, they can also be interpreted as describing the steady flow of a fluid.*

We shall be mainly concerned with the second theorem. Let

$$(2) \qquad \mathrm{K}f = u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$$

be the infinitesimal transformation of a one-parameter group G, where u, v, w are arbitrary functions of x, y, z. The finite equations of the group are found by integrating

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$$(3) \qquad \frac{dx}{dt} = u(x, y, z), \quad \frac{dy}{dt} = v(x, y, z), \quad \frac{dz}{dt} = w(x, y, z),$$

with the condition $x = a$, $y = b$, $z = c$ for $t = 0$, if a, b, c are the variables which are transformed into x, y, z by the operations of the group. Thus u, v, w are the values of the velocity-components at the point x, y, z.

Let X, Y, Z denote the components of the force acting upon the unit of mass, p the pressure, and ρ the density at the point x, y, z. Then EULER'S equations assume the form:

$$(4) \qquad \mathrm{K}u = X - \frac{1}{\rho}\frac{\partial p}{\partial x}, \quad \mathrm{K}v = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}, \quad \mathrm{K}w = Z - \frac{1}{\rho}\frac{\partial p}{\partial z},$$

and the equation of continuity becomes:

$$(4a) \qquad \mathrm{K}\rho + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0.$$

Moreover p and ρ are connected by an equation:

$$(4b) \qquad p = f(\rho),$$

where the nature of the function $f(\rho)$ depends upon the character of the fluid considered.

Suppose that u, v, w are arbitrarily assigned functions of x, y, z. Then by integrating (3) we get x, y, z as functions of a, b, c, and t. The equations (1), thus found, represent any one-parameter group G in space. We can find the physical conditions corresponding to every group G. First ρ may be found by integrating (4a). The simplest way to do this is to consider ρ as a function of a, b, c and t rather than of x, y, z. For then, since the motion is steady, i. e., since $\partial\rho/\partial t = 0$, $\mathrm{K}\rho$ is simply $d\rho/dt$, and therefore (4a) may be written:

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = -\sigma(a, b, c; t),$$

whence

$$(5) \qquad \rho = \rho_0 e^{-\int_0^t \sigma(a, b, c; t)dt},$$

where ρ_0 denotes the value of ρ for $t = 0$. In the case of steady motion, ρ_0 is not an arbitrary function of a, b, c over any region of space. For if it were, ρ would be the same function of x, y, z that ρ_0 is of a, b, c and therefore ρ would be an arbitrary function of x, y, z which according to (4a) is obviously not the case. The reason for this difficulty is that although we have assumed explicitly that u, v, w are functions of x, y, z alone, we have as yet made no explicit assumption of the same character about ρ, equation (5) being universally true whether the motion be stationary or not.

If S be a surface, cutting all of the lines of flow once and only once, ρ can obviously be assigned arbitrary values for all points upon this surface. Of course S may consist of several separate pieces. When any element of the fluid reaches this surface, its density must then, if the motion is stationary, assume the value corresponding to that point of S. Thus and only thus can the motion be stationary.

After ρ is obtained ($4b$) gives p, and from (4), X, Y, Z can be found. If u, v, w are real, and ρ_0 is positive, ρ from (5) is positive as it should be. If p should become negative, the fluid would tear.

In the case of an incompressible fluid, these results would be somewhat different: p is not then a function of ρ. We then have

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{6}$$

so that u, v, w must be taken subject to this condition. The surfaces $\rho =$ constant are made up of lines of flow, for the density of an element of an incompressible fluid can not change. From one line of flow to another the density can vary according to an arbitrary law, for ($4a$) has reduced to $\mathrm{K}\rho = 0$, which says only that the surfaces $\rho =$ constant are invariant under all transformations of the group, i. e., that they are made up of lines of flow, which we have already assumed to be the case. Since p is not in general a function of ρ, in this case p can be assumed to be an arbitrary function of x, y, z and then X, Y, Z are determined from (4).

If the fluid is homogeneous, without being incompressible, $\rho =$ constant, and then from ($4a$), since $\mathrm{K}\rho = 0$, we get

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

i. e., although the fluid may not be incompressible, if it is homogeneous and has a steady motion, it moves just as an incompressible fluid of like density would. But if the fluid is not incompressible, p is a certain function of ρ, and since $\rho =$ constant, $p =$ constant all through the fluid. The forces X, Y, Z are again obtained from (4).

The one-parameter group, generated by $\mathrm{K}f$, has two families of invariant surfaces, viz., the ∞^2 surfaces:

$$u_1(x, y, z) = \text{constant}, \quad u_2(x, y, z) = \text{constant},$$

where u_1, u_2 are two independent solutions of $\mathrm{K}f = 0$. The intersections of these surfaces are the lines of flow of the fluid.

Like all one-parameter groups, ours can be written in the form:

$$\Omega_i(x, y, z) = \Omega_i(a, b, c)\ (i=1, 2), \quad W(x, y, z) = W(a, b, c) + t. \tag{7}$$

Thus all steady fluid motions can be put into this form, and every set of equations of this kind represents a steady fluid motion.

It should be remarked, however, that it is not necessary that the functions Ω_i and W should be represented by the same formula for all portions of space. Thus, we have more generally

$$\begin{aligned} \Omega_{ki}(x, y, z) &= \Omega_{ki}(a, b, c) && (i=1, 2), \\ W_k(x, y, z) &= W_k(a, b, c) + t && (k=1, 2, 3, \cdots), \end{aligned} \tag{8}$$

the formulæ with index k being valid only for a certain region R_k of space.

Thus while the point x, y, z is in R_1, Ω_{1i} and W_1 are used, when it is in R_2, the next adjoining region, Ω_{2i} and W_2 are employed, etc. Of course, for continuity, the values of x, y, z obtained from the first set of equations must coincide with those found from the second set for all points upon the boundary between R_1 and R_2, i. e., x, y, z must be continuous functions of a, b, c, t. The same is not necessary of u, v, w, i. e., in general the direction of the motion will undergo a discontinuous change on the bounding surface between two such regions. Such discontinuities will actually occur, for instance, if we consider the flow of a fluid in a channel whose walls are portions of analytical surfaces, which meet at angles different from 180°.

These considerations go a great way towards the solution of the general problem of the steady flow of a fluid in a channel bounded by any arbitrary surfaces whatsoever. An infinity of equations of the form (8) can be written down in every case, which fulfil the boundary conditions, and which represent a possible fluid motion. Among all these, it will be necessary to pick out those which correspond to the external forces at work, i. e., which give the values to X, Y, Z belonging to that particular problem.

In most cases the forces X, Y, Z have a potential, so that

$$X = \frac{\partial V}{\partial x}, \quad Y = \frac{\partial V}{\partial y}, \quad Z = \frac{\partial V}{\partial z}. \tag{9}$$

In all such cases u, v, w cannot be chosen at will, but are subject to the condition, that the values of X, Y, Z computed from (4) can be written in the form (9). If we put

$$P = \int \frac{dp}{\rho}, \tag{10}$$

equations (4) become:

$$\mathrm{K}u = \frac{\partial(V-P)}{\partial x}, \quad \mathrm{K}v = \frac{\partial(V-P)}{\partial y}, \quad \mathrm{K}w = \frac{\partial(V-P)}{\partial z}, \tag{11}$$

and therefore, *if the forces X, Y, Z have a potential,*

$$\mathrm{K}u\,dx + \mathrm{K}v\,dy + \mathrm{K}w\,dz = d(V - P) \tag{12}$$

must be a complete differential. This is the condition which u, v, w must verify in this case, or, what amounts to the same thing,

$$\frac{\partial \mathrm{K}v}{\partial z} - \frac{\partial \mathrm{K}w}{\partial y} = 0, \quad \frac{\partial \mathrm{K}w}{\partial x} - \frac{\partial \mathrm{K}u}{\partial z} = 0, \quad \frac{\partial \mathrm{K}u}{\partial y} - \frac{\partial \mathrm{K}v}{\partial x} = 0. \tag{13}$$

§ 2. *The fluid motion expressed by the general projective group.*

We shall confine ourselves to the case in which the forces have a potential and we shall find that the most general ternary projective group, which can express the steady motion of a fluid, if the forces have a potential, is a linear group.

For convenience in this paragraph we shall write x_1, x_2, x_3, u_1, u_2, u_3 instead of x, y, z, u, v, w.

The most general projective infinitesimal transformation is a linear combination of the following fifteen:

$$p_i = \frac{\partial f}{\partial x_i}, \quad x_i p_k = T_{ik}, \quad x_i \sum_{k=1}^{3} x_k p_k = P_i \qquad (i, k = 1, 2, 3).$$

We can write it:

$$\mathrm{K}f = \sum_{i=1}^{3} u_i \frac{\partial f}{\partial x_i}, \tag{14}$$

where

$$u_i = \gamma_i + \sum_{k=1}^{3} c_{ki} x_k + x_i \sum_{k=1}^{3} c_k x_k \qquad (i = 1, 2, 3). \tag{15}$$

We must find the necessary and sufficient conditions in order that

$$\sum_{i=1}^{3} \mathrm{K}u_i\,dx_i$$

may be a complete differential.

We obtain

$$v_i = \mathrm{K}u_i = (c_1 x_1 + c_2 x_2 + c_3 x_3) u_i + \sum_{k=1}^{3} (c_{ki} + c_k x_i) u_k \qquad (i = 1, 2, 3). \tag{16}$$

But if the forces have a potential, equations (13) of § 1 must be verified, the first of which is

$$\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} = 0.$$

We find

$$\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} = c_2 u_3 - c_3 u_2 + (c_1 x_1 + c_2 x_2 + c_3 x_3)\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + \sum_{k=1}^{3} \left((c_{k3} + c_k x_3) \frac{\partial u_k}{\partial x_2} - (c_{k2} + c_k x_2) \frac{\partial u_k}{\partial x_3} \right).$$

Now, from (2), we find

$$\frac{\partial u_i}{\partial x_j} = c_{ji} + c_j x_i + \delta_{ij} \sum_{\lambda=1}^{3} c_\lambda x_\lambda ,$$

where $\delta_{ij} = 0$ for $i + j$, and $\delta_{ii} = 1$.

Thus we obtain:

$$\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} = c_2 u_3 - c_3 u_2 + [c_{23} - c_{32} + (c_2 x_3 - c_3 x_2)] \sum_{\lambda=1}^{3} c_\lambda x_\lambda$$
$$+ \sum_{k=1}^{3} [(c_{k3} + c_k x_3)(c_{2k} + c_2 x_k + \delta_{k2} \sum_{\lambda=1}^{3} c_\lambda x_\lambda) - (c_{k2} + c_k x_2)(c_{3k} + c_3 x_k + \delta_{k3} \sum_{\lambda=1}^{3} c_\lambda x_\lambda)] .$$

When this is arranged according to powers of x_1, x_2, x_3, the coefficient of each term must vanish. This gives the following equations, obtained by putting the coefficients of the quadratic terms equal to zero:

$$c_2 c_3 = c_3 c_1 = c_1 c_2 = c_2^2 - c_3^2 = 0 ,$$

which gives at once $c_2 = c_3 = 0$. Using one of the other equations (13) §1, we should find $c_1 = 0$ also.

Equating the other coefficients in $(\partial v_2/\partial x_3) - (\partial v_3/\partial x_2)$ to zero, we get only one other condition, and thus we have proved the theorem:

An arbitrary projective one-parameter group cannot represent the steady motion of a fluid under the influence of forces possessing a potential. To represent such a motion, the group must be a linear group. If

$$\mathrm{K}f = \sum_{k=1}^{3} u_i \frac{\partial f}{\partial x_i}, \tag{17}$$

where

$$u_i = \gamma_i + \sum_{k=1}^{3} c_{ki} x_k , \tag{18}$$

is the infinitesimal transformation of this linear group, the coefficients c_{ki} must verify the three conditions:

$$\begin{cases} \sum_{k=1}^{3} (c_{k3} c_{2k} - c_{k2} c_{3k}) = 0 , \\ \sum_{k=1}^{3} (c_{k1} c_{3k} - c_{k3} c_{1k}) = 0 , \\ \sum_{k=1}^{3} (c_{k2} c_{1k} - c_{k1} c_{2k}) = 0 , \end{cases} \tag{19}$$

which necessary conditions are also sufficient.

§ 3. *Fluid motion expressed by the ternary linear group.*

In equations (18) we can put $\gamma_i = 0$ without loss of generality. For the motions in which $\gamma_i \neq 0$ only differ from those for which $\gamma_i = 0$ by a translation.

We assume then that the origin remains fixed during the entire motion, and we will resume the less symmetrical but more convenient notation, x, y, z: u, v, w instead of x_1, x_2, x_3; u_1, u_2, u_3. Let

$$(20) \qquad \begin{cases} u = a_1 x + b_1 y + c_1 z, \\ v = a_2 x + b_2 y + c_2 z, \\ w = a_3 x + b_3 y + c_3 z. \end{cases}$$

The components of the velocity of rotation of any element of the fluid are:

$$\begin{cases} \xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \frac{1}{2}(b_3 - c_2), \\ \eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \frac{1}{2}(c_1 - a_3), \\ \zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(a_2 - b_1), \end{cases}$$

so that ξ, η, ζ are the same for all points of the fluid. Hence at all points the axis of rotation has the same direction, and the angular velocity of rotation is the same, viz.:

$$\omega = \sqrt{\xi^2 + \eta^2 + \zeta^2}.$$

Let us take our z axis parallel to this direction. Then

$$\xi = \eta = 0, \quad \zeta = \omega,$$

and hence, for this choice of coördinates,

$$b_3 = c_2, \quad c_1 = a_3, \quad a_2 = b_1 + 2\omega,$$

so that we can write more simply

$$(21) \qquad \begin{cases} u = \quad Ax + (H - \omega)y + Gz, \\ v = (H + \omega)x + \quad By + Fz, \\ w = \quad Gx + \quad Fy + Cz. \end{cases}$$

If the forces have a potential, equations (19) give

$$(A + B)\omega = 0, \quad F\omega = 0, \quad G\omega = 0.$$

Therefore, either $\omega = 0$, or

$$A + B = 0, \quad F = 0, \quad G = 0.$$

If $\omega = 0$, we have a velocity potential

$$\phi = \tfrac{1}{2}(Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gxz + 2Hxy),$$

so that the lines of flow are the orthogonal trajectories of the system of similar surfaces of the second order:

$$\phi = \text{constant}.$$

In the second case (21) reduces to

$$(22) \qquad \begin{cases} u = & Ax + (H-\omega)y, \\ v = (H+\omega)x - & Ay, \\ w = & Cz. \end{cases}$$

The motion can be decomposed into a rotation and a motion with a velocity potential. But it is simpler to treat it without so decomposing it. This case again separates into two distinct sub-cases.

In the first sub-case,

$$A^2 + H^2 - \omega \neq 0.$$

Then, if

$$(23) \qquad \begin{cases} \rho_1 = +\sqrt{A^2 + H^2 - \omega} = \rho, \quad \rho_2 = -\rho_1 = -\rho, \\ \lambda_i = -\dfrac{H-\omega}{A-\rho_i} = \dfrac{A+\rho_i}{H+\omega}, \end{cases}$$

the finite equations of the group can be written as follows:

$$(24) \qquad x - \lambda_1 y = e^{-\rho t}(a - \lambda_1 b), \quad x - \lambda_2 y = e^{+\rho t}(a - \lambda_2 b), \quad z = ce^{Ct},$$

where a, b, c are the values of x, y, z for $t = 0$.

The surfaces:

$$z^{-\rho}(x - \lambda_1 y)^{-C} = \text{constant}, \qquad z^{\rho}(x - \lambda_2 y)^{-C} = \text{constant},$$

which are real or imaginary cylinders, are therefore invariant. Their real intersections are the lines of flow.

The function

$$(x - \lambda_1 y)(x - \lambda_2 y)$$

is also invariant. Therefore the projections of the lines of flow upon the xy plane are the similar conics:

$$(25) \qquad x^2 - \frac{2A}{H+\omega}xy + \frac{\omega - H}{\omega + H}y^2 = \text{constant},$$

i. e., the lines of flow are situated upon a cylinder of the second order, whose elements are parallel to the z axis.

The physical conditions, corresponding to this motion, are obtained from the considerations of § 1. We have, in particular, from (11),

$$(26) \qquad V - p = \tfrac{1}{2}\left[(A^2 + H^2 - \omega)(x^2 + y^2) + c^2 z^2\right] + \text{constant}.$$

The motion, here considered, becomes periodic if ρ is purely imaginary, and $c = 0$, as may be seen by solving (24) for x, y, z. The lines of flow are then similar ellipses in parallel planes. It can be shown that this motion is possible for an incompressible homogeneous fluid filling an ellipsoidal space, the particles attracting each other according to Newton's law. This has been studied in detail by DIRICHLET and DEDEKIND.* It can also be easily shown that the analogous result is true for an elliptic cylinder of infinite length, provided that H and ω satisfy the equation:

$$(H^2 - \omega^2)\frac{H^2 + 4}{\pi f \rho} + 4 = 0,$$

where ρ is the density, and f a numerical constant, depending upon the units employed. If a' and b' are the semi-axes of the elliptic section of the cylinder their ratio will be obtained from the equation:

$$\frac{a'}{b'} = \sqrt{\frac{\omega - H}{\omega + H}}.$$

In the second sub-case,

$$\begin{cases} A^2 + H^2 - \omega = 0, \\ \lambda = -\dfrac{H - \omega}{A} = \dfrac{A}{H + \omega}. \end{cases}$$

The finite equations of the group can be written:

$$x - \lambda y = a - \lambda b, \quad \frac{y}{x - \lambda y} = \frac{b}{a - \lambda b} + (H + \omega)t, \quad z = ce^{\sigma}.$$

If we chose the plane $x - \lambda y = 0$ as xz-plane, and the plane $\lambda x + y = 0$ as yz-plane, the planes $y =$ constant are invariant, and the lines of flow are the curves:

$$z = \text{constant } e^{\frac{\sigma}{2\omega}\frac{x}{y}}, \quad y = \text{constant},$$

i. e., exponential curves in these planes.

§ 4. *Fluid motion expressed by a linearoid group.*

Let us consider next a one-parameter group, generated by

$$\mathrm{K}f = u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z},$$

* Crelle's Journal, vol. 58.

where

$$(27)\qquad \begin{cases} u = \phi_1 x + \phi_2 y + \phi_3, \\ v = \psi_1 x + \psi_2 y + \psi_3, \\ w = \chi_1 x + \chi_2 y + \chi_3, \end{cases}$$

$\phi_1, \cdots, \chi_3$ being functions of z only. It will be of the class called linearoid by me in another connection. Then,

$$(28)\qquad \begin{cases} \mathrm{K}u = (\phi_1^2 + \phi_2\psi_1 + \phi_3'\chi_1)x + (\phi_1\phi_2 + \phi_2\psi_2 + \phi_3'\chi_2)y + \phi_1\phi_3 + \phi_2\psi_3 \\ \qquad\qquad + \phi_3'\chi_3 + \phi_1'\chi_1 x^2 + (\phi_1'\chi_2 + \phi_2'\chi_1)xy + \phi_2'\chi_2 y^2, \\ \mathrm{K}v = (\psi_1\phi_1 + \psi_2\psi_1 + \psi_3'\chi_1)x + (\psi_1\phi_2 + \psi_2^2 + \psi_3'\chi_2)y + \psi_1\phi_3 + \psi_2\psi_3 \\ \qquad\qquad + \psi_3'\chi_3 + \psi_1'\chi_1 x^2 + (\psi_1'\chi_2 + \psi_2'\chi_1)xy + \psi_2'\chi_2 y^2, \\ \mathrm{K}w = (\chi_1\phi_1 + \chi_2\psi_1 + \chi_3'\chi_1)x + (\chi_1\phi_2 + \chi_2\psi_2 + \chi_3'\chi_2)y + \chi_1\phi_3 + \chi_2\psi_3 \\ \qquad\qquad + \chi_3'\chi_3 + \chi_1'\chi_1 x^2 + (\chi_1'\chi_2 + \chi_2'\chi_1)xy + \chi_2'\chi_2 y_2, \end{cases}$$

where the accents denote derivatives with respect to z.

Write this as follows:

$$(29)\qquad \begin{cases} \mathrm{K}u = \lambda_1 x^2 + 2\mu_1 xy + \nu_1 y^2 + 2\rho_1 x + 2\sigma_1 y + 2\tau_1, \\ \mathrm{K}v = \lambda_2 x^2 + 2\mu_2 xy + \nu_2 y^2 + 2\rho_2 x + 2\sigma_2 y + 2\tau_2, \\ \mathrm{K}w = \lambda_3 x^2 + 2\mu_3 xy + \nu_3 y^2 + 2\rho_3 x + 2\sigma_3 y + 2\tau_3; \end{cases}$$

the conditions for the integrability of

$$\mathrm{K}u \cdot dx + \mathrm{K}v \cdot dy + \mathrm{K}w \cdot dz,$$

i. e., for the existence of a potential for the corresponding fluid motion, become:

$$(30)\qquad \begin{cases} \mu_1 - \lambda_2 = 0, \quad \nu_1 - \mu_2 = 0, \quad \sigma_1 - \rho_2 = 0, \\ \rho_2' - \mu_3 = 0, \quad \sigma_2' - \nu_3 = 0, \quad \tau_2' - \sigma_3 = 0, \\ \rho_1' - \lambda_3 = 0, \quad \sigma_1' - \mu_3 = 0, \quad \tau_1' - \rho_3 = 0, \\ \lambda_1' = \mu_1' = \nu_1' = \lambda_2' = \mu_2' = \nu_2' = 0. \end{cases}$$

Hence

$$(31)\qquad \begin{cases} 2(\mu_1 - \lambda_2) = (\phi_2' - 2\psi_1')\chi_1 + \phi_1'\chi_2 = 0, \\ 2(\nu_1 - \mu_2) = -\psi_2'\chi_1 + (2\phi_2' - \psi_1')\chi_2 = 0, \end{cases}$$

so that we have two cases to distinguish, according as $\chi_1 = \chi_2 = 0$, or the determinant

$$(32)\qquad (\phi_2' - 2\psi_1')(2\phi_2' - \psi_1') + \phi_1'\psi_2' = 0.$$

Let us consider first case I, in which $\chi_1 = \chi_2 = 0$. Then we find from the definition of these quantities,

$$\lambda_1 = \mu_1 = \nu_1 = \lambda_2 = \mu_2 = \nu_2 = \lambda_3 = \mu_3 = \nu_3 = \rho_3 = \sigma_3 = 0,$$

and from (30),

$$\rho_1' = \rho_2' = \sigma_1' = \sigma_2' = \tau_1' = \tau_2' = 0,$$

so that ρ_1, ρ_2, σ_1, σ_2, τ_1, τ_2 are constants.

Now since $\chi_1 = \chi_2 = 0$, we have

$$(33) \qquad \begin{cases} 2\rho_1 = \phi_1^2 + \phi_2\psi_1, & 2\rho_2 = \psi_1(\phi_1 + \psi_2), \\ 2\sigma_1 = \phi_2(\phi_1 + \psi_2), & 2\sigma_2 = \phi_2\psi_1 + \psi_2^2. \end{cases}$$

The only further condition is $\sigma_1 = \rho_2$, whence either

$$\phi_1 + \psi_2 = 0, \qquad \text{or} \qquad \phi_2 = \psi_1.$$

We will first assume that $\phi_1 + \psi_2 = 0$, and call this case Ia. Then we have, in this case,

$$\sigma_1 = \rho_2 = 0, \qquad \phi_1^2 + \phi_2\psi_1 = 2\rho_1 = 2\sigma_2 = 2\sigma,$$

where σ is a constant. Thus in case Ia the functions χ_1 and χ_2 vanish, and between the other functions we have the relations:

$$(34) \qquad \begin{cases} \phi_1 + \psi_2 = 0, \quad \phi_1^2 + \phi_2\psi_1 = 2\sigma = \text{constant}, \\ \phi_1\phi_3 + \phi_2\psi_3 + \chi_3\phi_3' = 2\tau_1, \\ \psi_1\phi_3 + \psi_2\psi_3 + \chi_3\psi_3' = 2\tau_2, \end{cases}$$

where τ_1, τ_2 are also constants, and where ϕ_1, ϕ_2, χ_3 may be taken as arbitrary functions of z.

In the second case, case Ib, we have

$$\phi_2 = \psi_1,$$

whence

$$(35) \qquad \begin{cases} 2\rho_2 = 2\sigma_1 = \phi_2(\phi_1 + \psi_2), \\ 2\rho_1 = \phi_2^2 + \phi_2^2, \quad 2\sigma_2 = \phi_2^2 + \psi_2^2, \end{cases}$$

where ϕ_1 and ψ_2 may be taken to be arbitrary functions of z, and the equations for ϕ_3, ψ_3, χ_3 are of the same form as in (34).

If the fluid is incompressible we have further, $\chi_3 =$ constant in case Ia, and in case Ib, we find

$$\chi_3 = -\int (\phi_1 + \psi_2)dz + \text{constant},$$

as is seen at once from the condition of incompressibility, viz.:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0\,.$$

We shall not at present further discuss the fluid motions thus analytically determined. *The group theoretical considerations have enabled us to discover a considerable class of fluid motions, whose entire theory depends only on linear differential equations.*

Let us now carry out the corresponding investigation for case II, in which χ_1 and χ_2 do not both vanish. We have then

$$(\phi_2'-2\psi_1')(2\phi_2'-\psi_1')+\phi_1'\psi_2'=0\,, \tag{32}$$

and further, as before (equations (30)) λ_1, μ_1, ν_1, λ_2, μ_2, ν_2 are constants. By the definition of these quantities, we have

$$\begin{cases}\phi_1'\chi_1=\lambda_1\,, & \phi_2'\chi_1+\phi_1'\chi_2=2\mu_1\,, & \phi_2'\chi_2=\nu_1\,,\\ \psi_1'\chi_1=\lambda_2\,, & \psi_2'\chi_1+\psi_1'\chi_2=2\mu_2\,, & \psi_2'\chi_2=\nu_2\,.\end{cases} \tag{36}$$

Therefore

$$\nu_i\chi_1^2+\lambda_i\chi_2^2-2\mu_i\chi_1\chi_2=0 \qquad (i=1,\,2), \tag{37}$$

and hence

$$\lambda_1:2\mu_1:\nu_1=\lambda_2:2\mu_2:\nu_2=\alpha:\beta:1\,, \tag{38}$$

where α and β denote the ratios $\lambda_1:\nu_1$ and $2\mu_1:\nu_1$.

Multiplying (32) by $\chi_1^2\chi_2^2$, and using (36) and (38), we obtain a third quadratic for χ_1 and χ_2, viz.:

$$(\nu_1\chi_1-\lambda_2\chi_2)^2=0\,, \tag{39}$$

so that also

$$\lambda_2^2:\,2\lambda_2\nu_1:\,\nu_1^2=\alpha:\,\beta:\,1.$$

Now if we rotate the x axis through an angle θ, the form of u, v, w is not changed, and w becomes

$$\chi_1(x\cos\theta-y\sin\theta)+\chi_2(x\sin\theta+y\cos\theta)+\chi_3$$
$$=(\chi_1\cos\theta+\chi_2\sin\theta)\,x+(-\chi_1\sin\theta+\chi_2\cos\theta)\,y+\chi_3\,.$$

If we put

$$\tan\theta=\frac{\nu_1}{\lambda_2}\,,$$

according to (39), the coefficient of y in the transformed w will vanish.

We can therefore assume from the beginning, without loss of generality, that $\chi_2 = 0$. Having already treated the case $\chi_1 = \chi_2 = 0$, we will now assume $\chi_1 \neq 0$, $\chi_2 = 0$.

From (36) we have, in this case,

$$\begin{cases} \phi_1' = \dfrac{\lambda_1}{\chi_1}, & \phi_2' = \dfrac{2\mu_1}{\chi_1}, & \nu_1 = 0, \\ \psi_1' = \dfrac{\lambda_2}{\chi_1}, & \psi_2' = \dfrac{2\mu_2}{\chi_1}, & \nu_2 = 0. \end{cases}$$

Since $\nu_2 = 0$, (30) gives $\mu_2 = 0$. Therefore, if

$$\int \frac{dz}{\chi_1} = \phi,$$

we have, since $\mu_1 = \lambda_2$,

$$(40) \qquad \begin{cases} \phi_1 = \lambda_1\phi + \alpha, & \phi_2 = 2\lambda_2\phi + \beta, \\ \psi_1 = \lambda_2\phi + \gamma, & \psi_2 = \delta, \end{cases}$$

where α, β, γ, δ denote four arbitrary constants.

We find further,

$$\lambda_3 = \chi_1'\chi_1, \qquad \mu_3 = 0, \qquad \nu_3 = 0.$$

Therefore, from (30), we notice that ρ_2, σ_2, σ_1 are constants. Now we have, with the values for ϕ_1, ϕ_2, ψ_1, ψ_2 just found,

$$2\sigma_2 = 2\lambda_2^2\phi^2 + \lambda_2(\beta + 2\gamma)\phi + \beta\gamma + \delta^2,$$

so that λ_2 must vanish. Then

$$2\sigma_1 = \beta(\lambda_1\phi + \alpha + \delta),$$

so that λ_1 must also vanish. From the definition of ρ_2, which must be a constant, follows:

$$(41) \qquad \psi_3' = \frac{2\rho_2 - \gamma(\alpha + \delta)}{\chi_1} = \frac{(\beta - \gamma)(\alpha + \delta)}{\chi_1},$$

since $\rho_2 = \sigma_1$. Similarly we find, from $\rho_1' = \lambda_3 = \chi_1'\chi_1$,

$$(42) \qquad \phi_3' = \chi_1 + \frac{\epsilon - \alpha^2 - \beta\gamma}{\chi_1},$$

where ϵ denotes another constant.

Substituting all these values into the only two equations (30) not yet used, viz.: $\tau_2' = \sigma_3$, $\tau_1' = \rho_3$, we find the following conditions for χ_1 and χ_3:

$$(43)\quad \begin{cases} (\gamma-\beta)\chi_1 + \dfrac{\gamma(\epsilon - a^2 - \beta\gamma) + \delta(\beta-\gamma)(a+\delta)}{\chi_1} \\ \qquad\qquad + (\beta-\gamma)(a+\delta)\dfrac{d}{dz}\left(\dfrac{\chi_3}{\chi_1}\right) = 0\,, \\ \dfrac{a(\epsilon - a^2 - \beta\gamma) + \beta(\beta-\gamma)(a-\delta)}{\chi_1} + \dfrac{d}{dz}(\chi_1\chi_3) \\ \qquad\qquad + (\epsilon - a^2 - \beta\gamma)\dfrac{d}{dz}\left(\dfrac{\chi_3}{\chi_1}\right) - \chi_1\chi_3' = 0\,. \end{cases}$$

The fluid motion, in this case, is therefore given by the equations:

$$(44)\quad \begin{cases} u = ax + \beta y + \phi_3\,, \\ v = \gamma x + \delta y + \psi_3\,, \\ w = \chi_1 x + \chi_3\,, \end{cases}$$

where a, β, γ, δ are constants, where χ_1 and χ_3 are obtained as functions of z by integrating (43), and where ϕ_3 and ψ_3 are determined from (41) and (42).

If the fluid is incompressible, we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = a + \delta + \chi_1' x + \chi_3' = 0\,,$$

i. e.,

$$\chi_1 = \eta\,, \quad \chi_3 = -(a+\delta)z + \theta\,,$$

where η and θ are constants. Moreover θ can be put equal to zero, if the xy-plane be appropriately chosen. Substituting these values in (43), we get relations between η and the other constants. But as will be seen from (41) and (42) the group in this case becomes a ternary linear one, all of the coefficients being constants.

Thus Case II of this paragraph gives nothing new for incompressible fluids.

The detailed discussion of the fluid motions here determined, as well as those of the preceding case, will be left for a future occasion.

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DETERMINATION OF AN ABSTRACT SIMPLE GROUP OF ORDER $2^7 \cdot 3^6 \cdot 5 \cdot 7$ HOLOEDRICALLY ISOMORPHIC WITH A CERTAIN ORTHOGONAL GROUP AND WITH A CERTAIN HYPERABELIAN GROUP*

BY

LEONARD EUGENE DICKSON

1. Among the known simple groups† occur an orthogonal group and a hyperabelian group of the same order $2^7 \cdot 3^6 \cdot 5 \cdot 7$. They are shown to be holoedrically isomorphic in this paper. We first determine in §§ 2–14 an abstract group Γ (§§ 4, 2) simply isomorphic with the orthogonal group. This is accomplished by means of a rectangular array, a direct method of procedure employed by the writer in two recent papers in the Proceedings of the London Mathematical Society (vol. 31, p. 30; vol. 31, p. 351).

§§ 2, 3. *Rectangular array for the orthogonal group $O_{6,3}$ with respect to the subgroup $O_{5,3}$.*

2. The general orthogonal group of senary linear substitutions of modulus 3 having determinant unity has a subgroup H of index two.‡ The group H has as maximal invariant subgroup the group of order two generated by the substitution C which changes the signs of the six indices. The quotient group is a simple group $O_{6,3}$ of order $2^7 \cdot 3^6 \cdot 5 \cdot 7$. The substitutions of H which affect only the indices $\xi_1, \cdots, \xi_5$ form a simple group $O_{5,3}$ of order $2^6 \cdot 3^4 \cdot 5$ and of index $2 \cdot 126$ under H.

In a paper communicated December 28, 1899, to the London Mathematical Society, the writer has shown that $O_{5,3}$ is simply isomorphic with the abstract group G generated by the operators E_1, E_2, E_3, B_1, W subject to the generational relations:

$$(1) \qquad E_1^3 = E_2^2 = E_3^2 = B_1^2 = W^3 = I,$$

$$(2) \qquad (E_1E_2)^3 = (E_2E_3)^3 = (B_1E_1)^3 = (E_1E_3)^2 = (B_1E_2)^2 = (B_1E_3)^2 = I,$$

$$(3) \qquad WE_1 = B_3E_2E_1W, \quad WE_2 = B_3W, \quad WB_1 = B_3E_2W,$$

* Presented to the Society (Chicago) April 14, 1900. Received for publication January 15, 1900.

† See the list of known simple groups in the Bulletin of the American Mathematical Society for July, 1899.

‡ American Journal of Mathematics, vol. 21 (1899), pp. 193–256.

$$(4) \qquad WB_4 = B_4B_2E_1E_2E_1^2W_1^2,$$

$$(5) \qquad WE_3E_2E_1WE_3 = E_1^2E_2E_3E_2E_1WE_3E_2E_1W,$$

where for brevity we have set

$$(6) \qquad B_2 \equiv E_1B_1E_1^2, \quad B_3 \equiv E_1E_2E_1^2B_1E_1E_2E_1^2, \quad B_4 \equiv E_2E_3B_3E_3E_2.$$

From the above relations we derive the following:

$$(7) \qquad E_2E_1W = B_3WE_1 = WE_2E_1,$$

$$(8) \qquad WE_1^2 = E_1^2WB_1, \quad E_1B_1B_2 = B_1E_1.$$

Indeed, we have

$$WE_1^2 \equiv WE_2E_1E_2E_1E_2 = E_2E_1E_2E_1WE_2 = E_1^2E_2 \cdot B_3W = E_1^2WB_1,$$

$$E_1B_1B_2 = E_1B_1 \cdot E_1B_1E_1^2 = (E_1B_1)^3B_1E_1 = B_1E_1.$$

The isomorphism is defined by the following correspondences:

$$(9) \quad E_1 \sim E_1' \equiv (\xi_1\xi_2\xi_3), \quad E_2 \sim E_2' \equiv (\xi_1\xi_2)(\xi_3\xi_4), \quad E_3 \sim E_3' \equiv (\xi_1\xi_2)(\xi_4\xi_5),$$

$$(10) \qquad W \sim W' \equiv \left\{ \begin{array}{l} \xi_1' = \xi_1 - \xi_2 - \xi_3 - \xi_4, \\ \xi_2' = \xi_1 - \xi_2 + \xi_3 + \xi_4 \\ \xi_3' = \xi_1 + \xi_2 - \xi_3 + \xi_4 \\ \xi_4' = \xi_1 + \xi_2 + \xi_3 - \xi_4 \end{array} \right\},$$

$$(11) \qquad B_1 \sim C_1C_2, \quad B_2 \sim C_2C_3, \quad B_3 \sim C_3C_4, \quad B_4 \sim C_4C_5,$$

where C_i denotes the orthogonal substitution changing the sign of ξ_i.

3. The group $O_{5,3}$ is extended to the orthogonal group H by the substitution

$$F' \equiv (\xi_1\xi_6)(\xi_2\xi_3).$$

The substitutions of H replace ξ_6 by every one of the $2 \cdot 126 \equiv 3^5 + 3^2$ linear functions *

$$(f) \qquad \sum_{i=1}^{6} a_i\xi_i \qquad \left[\sum_{i=1}^{6} a_i^2 \equiv 1 \pmod 3\right].$$

It follows that all the substitutions of H are given by the formula

$$S_iO_{5,3} \qquad (i = 1, 2, \cdots, 2 \cdot 126),$$

where the $2 \cdot 126$ substitutions S_i replace ξ_6 by the $2 \cdot 126$ distinct linear functions (f). In the quotient group $O_{6,3}$, S_i and $CS_i \equiv S_iC$ become identical. Denote by S_i' $(i = 1, \cdots, 126)$ the corresponding distinct substitutions of $O_{6,3}$.

* American Journal of Mathematics, l. c., foot of p. 195.

A rectangular array of the substitutions of $O_{6,3}$ is therefore given by the formula:

$$S'_i O_{5,3} \qquad (i = 1, 2, \cdots, 126).$$

To determine the S'_i, we note that the linear functions (f) are of the forms:

$$\pm \xi_i, \quad \pm \xi_i \pm \xi_j \pm \xi_k \pm \xi_l.$$

Since $-f$ is derived from f by the substitution C, we take only one of each pair $\pm f$ in determining the S'_i. For six of the S'_i we may take

$$(a) \qquad I, \; F', \; E'_2F', \; E'^2_1F', \; E'_2E'^2_1F', \; E'_3E'_2E'^2_1F'$$

which replace ξ_6 by ξ_6, ξ_1, ξ_2, ξ_3, ξ_4, ξ_5 respectively.

The substitution $W'F'$ replaces ξ_6 by $w \equiv \xi_1 - \xi_2 - \xi_3 - \xi_4$. Consider the 2^5 products K' of an even number of the $C_i\,(i = 1, \cdots, 6)$. If a particular K' replace w by ω, the four substitutions K', CK', C_5C_6K', $C_1C_2C_3C_4K'$ replace w by either ω or $-\omega$. Hence, of the 2^5 substitutions K', we need only consider 2^3 representatives, as

$$(k) \qquad \begin{gathered} K'_1 = I, \quad K'_i = C_1C_i\,(i = 2, \cdots, 5), \quad K'_6 = C_1C_2C_3C_5, \\ K'_7 = C_1C_2C_4C_5, \quad K'_8 = C_1C_3C_4C_5. \end{gathered}$$

The substitutions $K'_jW'F'$ may therefore be taken as eight of the S'_i, distinct from the above six.

Using for the moment the notation 1234 for $\xi_1 + \xi_2 + \xi_3 + \xi_4$, we find that the substitutions

$$(b) \qquad \begin{gathered} I, \; E'_3, \; E'_2E'_3, \; E'_1E'_2E'_3, \; E'^2_1E'_2E'_3, \; F', \; E'_2F', \; E'^2_1F', \; E'_2E'^2_1F', \\ E'_3F', \; E'_2E'_3F', \; E'_1E'_3F', \; E'^2_1E'_3F', \; E'_1E'_2E'_3F', \; E'_2E'^2_1E'_3F', \end{gathered}$$

respectively replace 1234 by 1234, 1235, 1245, 2345, 1345, 2346, 1346, 1246, 1236, 1356, 2456, 1256, 2356, 3456, 1456, giving each of the 15 combinations of 1, 2, 3, 4, 5, 6 four at a time.

It follows that we may take as our 126 substitutions S'_i the six substitutions (a) and the 120 obtained by multiplying the 16 substitutions (b) on the right hand by $K'_jW'F'$ $(j = 1, 2, \cdots, 8)$. We have therefore explained the origin of the table given in § 7. Indeed, from that table we obtain a rectangular array for $O_{6,3}$ by replacing the group G by $O_{5,3}$ and accenting all the letters.

§§ 4–14. *Determination of the abstract group* Γ.

4. Consider the abstract group Γ obtained by the extension of G by an operator F subject to the relations:

$$(12) \qquad F^2 = I, \quad (E_1F)^3 = I,$$

$$(13) \qquad B_1F = FB_1B_4,$$

$$E_2E_1F = FE_1^2E_2, \tag{14}$$

$$E_3E_1F = FE_3E_1, \tag{15}$$

$$WFWF = FWFW, \tag{16}$$

$$WV = VW, \quad V \equiv FE_3E_1^2E_2E_1E_3F \equiv V^{-1}, \tag{17}$$

$$FW^2B_1B_4FE_3WE_3FWF = B_1E_3WE_3. \tag{18}$$

We readily verify that the generators E_1', E_2', E_3', B_1', W', F' of $O_{6,3}$ satisfy the relations (12) $\cdots$ (18).* Note that $V \sim V' \equiv (\xi_2\xi_3)(\xi_5\xi_6)$. The order of Γ is therefore at least as great as the order of $O_{6,3}$. To complete the proof of their simple isomorphism it remains only to prove that the order of Γ is at most as great as the order of $O_{6,3}$.

5. From the relations (1), $\cdots$, (8), (12), $\cdots$, (18), we proceed to derive a number of relations needed below. From (14) and (15) we find

$$E_3E_1E_2E_1F = FE_3E_1 \cdot E_1^2E_2 = FE_3E_2. \tag{19}$$

Taking the reciprocal of (19) and multiplying on the right and left by F, we get

$$FE_2E_3 = E_1^2E_2E_1^2E_3F = E_2E_1E_3E_2F. \tag{20}$$

From (20) we find

$$E_1 = E_2FE_2E_3FE_3E_2. \tag{21}$$

From (12) and (14) we get

$$FE_1FE_2F = FE_1FE_1 \cdot E_1^2E_2F = FE_1FE_1FE_2E_1 = E_1^2E_2E_1. \tag{22}$$

From (12) and (7) we find

$$E_3E_2E_1^2FE_1E_2E_3 = E_3E_2E_1 \cdot FE_1^2F \cdot E_2E_3 = E_3FE_1^2E_2E_1^2 \cdot E_2E_1E_2E_3F$$
$$= E_1FE_3E_1 \cdot E_2E_1^2E_2E_3F = E_1FE_3E_1^2E_2E_1E_3F = E_1V:$$

$$E_1V = E_3E_2E_1^2FE_1E_2E_3. \tag{23}$$

$$E_1VE_1V \equiv (E_3E_2E_1^2FE_1E_2E_3)^2 = I. \tag{24}$$

Taking the reciprocal of (13) and applying $B_4^2 = \mathrm{I}$, $B_1B_4 = B_4B_1$, we find

$$FB_1 = B_4B_1F = B_4 \cdot FB_1B_4 = B_4FB_4B_1:$$

$$B_4F = FB_4. \tag{25}$$

* In regard to the relation corresponding to (18) it should be remarked that for orthogonal substitutions there occurs an additional factor C in one member.

Transforming (13) by E_3E_1, which by (15) transforms F into itself, we get

$$B_1B_2F = FB_1B_2B_4. \tag{26}$$

By (13), $B_1B_2F = B_2FB_1B_4$. Hence from (26) we find

$$B_2F = FB_2. \tag{27}$$

Applying (1), (2), (6), (8) and (14), we find *

$$E_2E_1B_1F = B_1B_2B_3E_2E_1F = B_1B_2B_3FE_1^2E_2,$$
$$E_2E_1FB_1B_4 = FE_1^2E_2B_1B_4 = FB_1B_3B_4E_1^2E_2.$$

Equating these products by virtue of (13), we find

$$B_1B_2B_3F = FB_1B_3B_4. \tag{28}$$

Combining (28) with (26) and (28) with (25), we find respectively

$$B_3F = FB_2B_3, \tag{29}$$

$$B_1B_2B_3B_4F = FB_1B_3. \tag{30}$$

Hence F transforms any product formed from B_1, B_2, B_3, B_4 into another such product.

6. Corresponding to the orthogonal substitutions K_i' defined by formulæ (k) of § 3, we have the following operators of Γ:

$$K_1 = I, \qquad K_2 = B_1, \qquad K_3 = B_1B_2, \quad K_4 = B_1B_2B_3,$$
$$K_5 = B_1B_2B_3B_4, \quad K_6 = B_1B_3B_4, \quad K_7 = B_1B_4, \quad K_8 = B_1B_2B_4.$$

Any product derived from E_1, E_2, E_3, B_1 transforms any product derived from B_1, B_2, B_3, B_4 into a product of the B_i, a statement made evident by considering the corresponding substitutions of the isomorphic group $O_{5,3}$. In virtue of the theorem at the end of § 5, a like result holds when the transformer is any operator (for example, V) derived from F, E_1, E_2, E_3, B_1.

Since $B_1B_3W = WB_1B_3$, we have on applying (30),

$$B_1B_3WFG = WFB_1B_2B_3B_4G = WFG.$$

Hence the products $BWFG$, where B runs through the 16 distinct products of the B_1, B_2, B_3, B_4, reduce to the eight distinct products K_jWFG ($j = 1, \cdots, 8$). For example, $B_2WFG = K_4WFG$.

* In establishing identities between operators of G it is frequently simpler to work with the corresponding substitutions of the simply isomorphic group $O_{5,3}$.

7. Consider the following set of operators belonging to Γ:

$$R_0 \equiv G \qquad R_1 \equiv FG \qquad R_2 \equiv E_2FG$$
$$R_3 \equiv E_1^2FG \qquad R_4 \equiv E_2E_1^2FG \qquad R_5 \equiv E_3E_2E_1^2FG$$
$$R_{1j} \equiv K_jWFG \qquad R_{2j} \equiv E_3K_jWFG \qquad R_{3j} \equiv E_2E_3K_jWFG$$
$$R_{4j} \equiv E_1E_2E_3K_jWFG \qquad R_{5j} \equiv E_1^2E_2E_3K_jWFG \qquad R_{6j} \equiv FK_jWFG$$
$$R_{7j} \equiv E_2FK_jWFG \qquad R_{8j} \equiv E_1^2FK_jWFG \qquad R_{9j} \equiv E_2E_1^2FK_jWFG$$
$$R_{10j} \equiv E_3FK_jWFG \qquad R_{11j} \equiv E_2E_3FK_jWFG \qquad R_{12j} \equiv E_1E_3FK_jWFG$$
$$R_{13j} \equiv E_1^2E_3FK_jWFG \qquad R_{14j} \equiv E_1E_2E_3FK_jWFG \qquad R_{15j} \equiv E_2E_1^2E_3FK_jWFG$$

where $j = 1, 2, \cdots, 8$.

From the developments given below it will follow that this table is a rectangular array of the operators of the abstract group Γ with G as first row and therefore that Γ *is holoedrically isomorphic with* $O_{6,3}$. Indeed, in §§ 10–14, we prove that the 126 rows of our table are merely permuted amongst themselves on applying as a left hand multiplier any of the generators E_1, E_2, E_3, B_1, W, F and therefore for an arbitrary operator of Γ. Since the row R_0 contains the identity, it will follow that an arbitrary operator of Γ belongs to the above table. The order of Γ is thus not greater than that of $O_{6,3}$.

8. Lemma I.—*The rows* R_{1j} $(j = 1, \cdots, 8)$ *are merely permuted upon applying as a left hand multiplier either* E_1 *or* E_2.

$$E_2R_{1j} \equiv E_2K_jWFG = K_iE_2WFG = K_iB_3WB_1FG$$
$$= K_tWFB_1B_4G = K_tWFG \equiv R_{1t}.$$
$$E_2E_1K_jWFG = K_rE_2E_1WFG = K_rWFE_1^2E_2G = K_rWFG \equiv R_{1r}.$$
$$E_1^2K_sWFG = E_2E_1 \cdot E_2E_1 \cdot E_2K_sWFG = K_jWFG,$$

by the preceding results. Hence

$$E_1R_{1j} \equiv E_1K_jWFG = K_sWFG \equiv R_{1s}.$$

9. Lemma II.—*The rows* R_{1j} *are permuted upon applying as a left hand multiplier the operator* V *defined by* (17).

$$VR_{1j} \equiv VK_jWFG = K_iVWFG \qquad \text{[by § 6]}$$
$$= K_iWVFG = K_iWFE_3E_1^2E_2E_1E_3 \qquad \text{[by (17)]}$$
$$= K_iWFG \equiv R_{1i}.$$

10. Theorem.—*The application of E_2 as a left hand multiplier permutes the* 126 *rows.*

By inspection we see that E_2 interchanges R_1 with R_2, R_3 with R_4, R_{2j} with R_{3j}, R_{6j} with R_{7j}, R_{8j} with R_{9j}, R_{10j} with R_{11j}, R_{13j} with R_{15j}. Furthermore,

$$E_2R_5 \equiv E_2E_3E_2E_1^2FG = E_3E_2E_3E_1^2FG = E_3E_2E_1^2E_3E_1FG$$
$$= E_3E_2E_1^2FE_3E_1G = E_3E_2E_1^2FG \equiv R_5 \qquad \text{[by use of (15)]}.$$
$$E_2R_{4j} \equiv E_2E_1E_2E_3K_jWFG = E_1^2E_2E_3E_1K_jWFG = R_{5t},$$

upon applying lemma I to replace E_1R_{1j} by R_{1t}.

The condition for the identity $E_2R_{12j} = R_{12t}$ is

$$E_2E_1E_3FK_jWFG = E_1E_3FK_tWFG,$$

or

$$FE_3E_1^2 \cdot E_2E_1E_3FK_jWFG = K_tWFG,$$

which is satisfied in virtue of lemma II.

Likewise, the condition for the identity $E_2R_{14j} = R_{14t}$ is

$$(E_1E_2E_3F)^{-1}E_2(E_1E_2E_3F)K_jWFG = K_tWFG,$$

which is satisfied by lemma II since we have

$$FE_3E_2E_1^2E_2E_1E_2E_3F = FE_3E_1^2E_2E_1E_3F \equiv V.$$

11. Theorem.—*The application of E_3 as a left hand multiplier permutes the* 126 *rows.*

By inspection E_3 interchanges R_4 with R_5, R_{1j} with R_{2j}, R_{6j} with R_{10j}, R_{8j} with R_{12j}.

$$E_3R_1 \equiv E_3FG = E_2FE_2E_1E_2E_3G = E_2FG \equiv R_2 \qquad \text{[by (21)]}.$$
$$E_3R_3 \equiv E_3E_1^2FG = E_1^2E_3E_1FG = E_1^2FE_3E_1G = E_1^2FG \equiv R_3.$$
$$E_3R_{3j} \equiv E_2E_3E_2K_jWFG = E_2E_3K_tWFG \equiv R_{3t} \qquad \text{[by lemma I]}.$$
$$E_3R_{4j} \equiv E_3E_1E_2E_3K_jWFG = E_1^2E_2E_3E_2K_jWFG$$
$$= E_1^2E_2E_3K_tWFG \equiv R_{5t} \qquad \text{[by lemma I]}.$$
$$E_3R_{7j} \equiv E_3E_2FK_jWFG = E_1^2E_3E_1^2E_2FK_jWFG$$
$$= E_1^2E_3FE_2E_1K_jWFG = E_1^2E_3FK_tWFG \equiv R_{13t},$$

upon applying (14) and lemma I.

The condition for the identity $E_3R_{9j} = R_{9t}$ is

$$(E_2E_1^2F)^{-1}E_3(E_2E_1^2F)K_jWFG = K_tWFG,$$

and it is satisfied in virtue of lemma II since

$$FE_1E_2E_3E_2E_1^2F = FE_1E_3E_2E_3E_1^2F = FE_3E_1^2E_2E_1E_3F \equiv V.$$

$$E_3R_{1j} \equiv E_3E_2E_3FK_jWFG = E_2E_3E_2FK_jWFG$$
$$= E_2E_3E_2 \cdot E_2E_1FK_iWFG \quad \text{[by lemma I and (14)]}$$
$$= E_2E_1^2E_3FK_iWFG = R_{15i}.$$

The condition for the identity $E_3R_{14j} = R_{14i}$ is

$$(E_1E_2E_3F)^{-1}E_3(E_1E_2E_3F)K_jWFG = K_iWFG,$$

and it is satisfied by lemmas I and II since we have

$$FE_3E_2E_1^2E_3E_1E_2E_3F = FE_3E_2E_1E_3E_2E_3F$$
$$= FE_3E_2E_1E_2E_3E_2F = FE_3E_1^2E_2E_1^2E_3E_2F$$
$$= FE_3E_1^2E_2E_1E_3 \cdot E_1^2E_2F = FE_3E_1^2E_2E_1E_3FE_2E_1 = VE_2E_1.$$

12. Theorem.—*The application of F as a left hand multiplier permutes the* 126 *rows.*

$$FR_0 = R_1.$$

$$FR_{1j} = R_{6j}.$$

$$FR_2 \equiv FE_2FG = E_1^2FE_1^2E_2E_1G = E_1^2FG \equiv R_3 \quad \text{[by (22)]}.$$

$$FR_4 \equiv FE_2E_1^2FG = E_1^2E_2FE_1FG \quad \text{[by (14)]}$$
$$= E_1^2E_2 \cdot E_1^2FE_1^2G = E_2E_1E_2FE_1^2G \quad \text{[by (12)]}$$
$$= E_2E_1^2E_1^2E_2FE_1^2G = E_2E_1^2FE_2G = R_4 \quad \text{[by (14)]}.$$

$$FR_5 \equiv FE_3E_2E_1^2FG = E_3E_2FE_2 \cdot FG \quad \text{[by (21)]}$$
$$= E_3E_2 \cdot FR_2 = E_3E_2R_3 \equiv E_3E_2E_1^2FG \equiv R_5.$$

$$FR_{2i} \equiv FE_3K_jWFG = E_3E_1FE_1^2K_jWFG \quad \text{[by (21)]}$$
$$= E_1^2E_3FK_iWFG \equiv R_{13i} \quad \text{[by lemma I]}.$$

The condition $FR_{3j} = R_{14i}$ or

$$(E_1E_2E_3F)^{-1}F(E_2E_3)K_jWFG = K_iWFG$$

is satisfied by lemma II, since we have, by (20),

$$FE_3E_2E_1^2FE_2E_3 = FE_3E_2E_1^2 \cdot E_1^2E_2E_1^2E_3F = FE_3E_1^2E_2E_1E_3F = V.$$

$$FR_{5j} \equiv FE_1^2E_2E_3K_jWFG = E_2E_3FK_iWFG \equiv R_{11i},$$

since by (14) and (15) we have

$$FE_1^2E_2E_3 = E_2E_1FE_3 = E_2E_1 \cdot E_3E_1FE_1^2 = E_2E_3FE_1^2 .$$

The condition for the identity $FR_{4j} = R_{4t}$ is satisfied by lemmas I and II:

$$(E_1E_2E_3)^{-1}FE_1E_2E_3K_jWFG \equiv E_1VK_jWFG = K_tWFG \quad \text{[by (23)]}.$$

$$FR_{7j} \equiv FE_2FK_jWFG = E_1^2FE_1^2E_2E_1K_jWFG \quad \text{[by (22)]}$$

$$= E_1^2FK_tWFG \equiv R_{8t} \quad \text{[by lemma I]}.$$

$$FR_{9j} \equiv FE_2E_1^2FK_jWFG = R_{9t} \equiv E_2E_1^2FK_tWFG ,$$

since

$$FE_2E_1^2F = E_1^2E_2FE_1F = E_1^2E_2 \cdot E_1^2FE_1^2$$

$$= E_2E_1^2E_1^2E_2FE_1^2 = E_2E_1^2FE_2E_1 \cdot E_1^2 .$$

$$FR_{10j} \equiv FE_3FK_jWFG = R_{12t} \equiv E_1E_3FK_tWFG ,$$

since

$$FE_3F = FE_3E_1 \cdot E_1^2F = E_3E_1 \cdot FE_1^2F = E_3E_1^2FE_1 = E_1E_3FE_1 .$$

The condition for the identity $FR_{15j} = R_{12k}$, viz.,

$$(E_2E_1^2E_3F)^{-1}F(E_2E_1^2E_3F)K_jWFG = K_kWFG ,$$

is seen to be satisfied in virtue of lemmas I and II as follows:

$$FE_3E_1E_2FE_2E_1^2E_3F = E_3E_1FE_2FE_2E_1^2E_3F = E_3 \cdot FE_1^2E_2E_1 \cdot E_2E_1^2E_3F$$

$$= E_1E_3E_1 \cdot FE_1E_2E_1E_3F \quad \text{[by (22)]}$$

$$= E_1FE_3E_1^2E_2E_1E_3F = E_1V .$$

13. By (21), E_1 is expressed as a product of the E_2, E_3, F. Hence by §§ 10–12, E_1 permutes the 126 rows when applied as a left hand multiplier.

From the following relations given under formulæ (8), (2), (13) above,

$$B_1E_1 = E_1B_1B_2 , \quad B_1E_2 = E_2B_1 , \quad B_1E_3 = E_3B_1 , \quad B_1F = B_1B_4 ,$$

and from the remarks in § 6 concerning the $K_j (j = 1, \cdots, 8)$, it follows that B_1 applied as a left hand multiplier permutes the 126 rows. Then by (6) a like result holds for B_2, B_3 and B_4.

14. Theorem.—*The application of W as a left hand multiplier permutes the* 126 *rows.*

$$WR_0 = R_0 .$$

$$WR_1 \equiv WFG \equiv R_{11} .$$

$$WR_2 \equiv WE_2FG = B_3WFG = B_1WB_1B_3FG = K_2WFG = R_{12} .$$

$$WR_3 \equiv WE_1^2FG = E_1^2WB_1FG = E_1^2WFG = K_3WFG \equiv R_{13}$$
[by lemma I].

$$WR_4 = WE_2E_1^2FG = E_2E_1WE_1FG = E_2E_1B_3E_2E_1WFG$$
$$= B_2E_2E_1 \cdot E_2E_1WFG = B_2WE_1^2E_2FG = K_4WFG \equiv R_{14}.$$

The condition for the identity $WR_5 = R_5$ is that $(E_3E_2E_1^2F)WE_3E_2E_1^2F$ shall belong to G. We shall verify that it equals $E_1E_2E_3E_2WE_2E_3E_2E_1^2$. The condition for this equality may be written

$$(E_1E_2E_3E_2)^{-1}FE_1E_2E_3WE_3E_2E_1^2E(E_2E_3E_2E_1^2)^{-1} = W.$$

By (23), the left member is equal to

$$E_2E_1VWE_1VE_2 = E_2E_1WVE_1VE_2 \qquad \text{[by (17)]}$$
$$= E_2E_1WE_1^2E_2 = W \qquad \text{[by (24) and (7)]}.$$

Each row R_{ij} is of the form AK_jWFG, where A denotes a product built from E_1, E_2, E_3, F. But E_1, E_2, E_3 each transform K_j into some K_i, a statement made evident by considering the isomorphic orthogonal substitutions. Furthermore, by § 6, F transforms K_j into some K_i. Hence each row R_{ij} may be given the form K_iAWFG.

In virtue of the following relations between orthogonal substitutions,

$$W'C_1C_2 = C_3C_4E_2''W', \qquad W'C_1C_3 = C_2C_4E_1'^2E_2'E_1'W',$$
$$W'C_1C_4 = C_2C_3E_1'E_2'E_1'^2W',$$
$$W'C_1C_5 = C_1C_5W'^2, \qquad W'C_1C_2C_3C_5 = C_1C_5E'E_2'E_1'^2W'^2,$$
$$W'C_1C_2C_4C_5 = C_1C_5E_1'^2E_2'E_1'W'^2, \qquad W'C_1C_3C_4C_5 = C_1C_5E_2'W'^2,$$

we have in the isomorphic group G the general relation

$$WK_i = K_sA'W^{\pm 1}$$

where A' is derived from E_1 and E_2. Hence

$$WR_{ij} = WK_iAWFG = K_sA'W^{\pm 1}AWFG.$$

But, by §§ 10 and 13, K_s or A' when applied as left hand multiplier permutes the 126 rows. Hence it remains only to prove that W (and hence also W^{-1}) permutes the rows $R_{i1} \equiv AWFG$ when applied as a left hand multiplier.

$$WR_{11} \equiv W^2FG = B_1B_2B_3B_4WB_1B_2B_3B_4FG = B_1B_2B_3B_4WFG$$
$$= K_5WFG = R_{15}.$$

$$WR_{61} \equiv WFWFG = FWFWG = FWFG = R_{61} \qquad \text{[by (16)]}.$$

The condition for the identity $WR_{21} = R_{33}$ is that the operator

$$(E_2E_3K_3WF)^{-1}WE_3WF \equiv F(W^2K_3E_3E_2WE_3W)F$$

shall belong to G. To it corresponds in the orthogonal group a substitution which corresponds to

$$S \equiv B_2B_3E_1E_2E_3E_2W^2E_3E_2E_1^2.$$

We proceed to prove that in the abstract group Γ:

$$FSF = W^2K_3E_3E_2WE_3W \equiv T_1.$$

Now

$$\begin{aligned}
FSF &= B_3FE_1E_2E_3\cdot E_2W^2E_3E_2E_1^2F \\
&= B_3E_1E_2E_3E_1V\cdot E_2W^2E_3E_2E_1^2F && \text{[by (23)]} \\
&= B_3E_1E_2E_3VW^2E_1^2E_2\cdot E_3E_2E_1^2F && \text{[by (24) and (7)]} \\
&= B_3E_1E_2E_3W^2E_3E_2E_1^2FE_1E_2E_3\cdot E_2E_3E_2E_1^2F && \text{[by (17) and (23)]} \\
&= B_3E_1E_2E_3W^2E_3E_2E_1^2FE_3E_1F \\
&= B_3E_1E_2E_3W^2E_3E_2E_1E_3 \equiv T_2 && \text{[by (15)].}
\end{aligned}$$

It remains to prove that the products denoted by T_1 and T_2 are equal. Each belongs to the group G; they will therefore be identical if the corresponding orthogonal substitutions are identical. But to both T_1 and T_2 there corresponds the same orthogonal substitution, viz.,

$$\begin{aligned}
\xi_1' &= \xi_1, \quad \xi_2' = -\xi_2 + \xi_3 - \xi_4 + \xi_5, \quad \xi_3' = \quad \xi_2 - \xi_3 - \xi_4 + \xi_5, \\
\xi_4' &= -\xi_2 - \xi_3 - \xi_4 - \xi_5, \quad \xi_5' = -\xi_2 - \xi_3 + \xi_4 + \xi_5.
\end{aligned}$$

The condition for the identity $WR_{101} = R_{107}$ is that the product

$$(E_3FB_1B_4WF)^{-1}WE_3FWF$$

shall belong to G. It is satisfied in virtue of relation (18).

In view of the following relations derived from (3) and (8),

$$WE_2 = B_3W, \quad WE_1 = B_3E_2E_1W, \quad WE_1^2 = E_1^2B_3E_2W,$$

WR_{31}, WR_{41} and WR_{51} are each of the form

$$DWE_3WFG \equiv DWR_{21} = DR_{33},$$

where D is derived from B_3, E_1, E_2. Also WR_{71}, WR_{81}, WR_{91} are each of the form

$$DWFWFG \equiv DWR_{61} = DR_{61}.$$

Finally, the products WR_{i1} ($i = 11, 12, 13, 14, 15$) are each of the form

$$DWE_3FWFG \equiv DWR_{101} = DR_{107}.$$

By the results of §§ 10 and 13, each of the products DR_{33}, DR_{61}, DR_{107} equals some row R_i or R_{ij}.

§§ 15–17. *Isomorphism and correspondence of generators between the orthogonal and hyperabelian groups.*

15. The simple group $H_{4,3}$ of order 25920, which is derived from the decomposition of the Abelian group of modulus 3 on four indices, is simply isomorphic with the simple subgroup $O_{5,3}$ of the quinary orthogonal group of modulus 3.* We proceed to determine the operators of the former group which correspond to the generators E_1', E_2', E_3', C_1C_2, W' of the latter. We first determine the operators of $O_{5,3}$ which correspond to the generators † $\bar{B}_1$, $\bar{B}_2$, $\bar{B}_3$, $\bar{B}_4$, $\bar{B}_5$, $\bar{B}$ of $H_{4,3}$ given on pages 65 and 67 of volume 31 of the Proceedings of the London Mathematical Society. Of the two possible forms for $\bar{B}$, we choose that one given by $\gamma_1 \equiv 1 \pmod 3$, viz.,

$$\bar{B} = \pm \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

By the general correspondence set up in §5 of the paper cited in the foot-note, we find that $\bar{B}$ corresponds to the substitution

(31)

	ξ_1	Y_{13}	Y_{14}	Y_{23}	Y_{24}
$\xi_1' =$	1	0	0	0	0
$Y_{13}' =$	0	1	0	1	0
$Y_{14}' =$	0	2	1	2	1
$Y_{23}' =$	0	0	0	1	0
$Y_{24}' =$	0	0	0	2	1

leaving invariant modulo 3 the function

$$\phi \equiv \xi_1^2 + Y_{13}Y_{24} - Y_{14}Y_{23}.$$

*Transactions of the American Mathematical Society, vol. 1, p. 95.

†The substitutions $\bar{B}_i$ enter §15 alone and are to be distinguished from the earlier B_i.

We introduce the new indices

$$\xi_2 \equiv -Y_{13} - Y_{24}, \qquad \xi_3 \equiv Y_{13} + Y_{14} + Y_{23} - Y_{24},$$
$$\xi_4 \equiv -Y_{13} + Y_{14} + Y_{23} + Y_{24}, \quad \xi_5 \equiv -Y_{14} + Y_{23}. \tag{32}$$

Then

$$\phi \equiv \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2 \pmod 3.$$

Solving modulo 3 the relations (32), we find

$$Y_{13} \equiv \xi_2 + \xi_3 - \xi_4, \quad Y_{14} \equiv \xi_3 + \xi_4 + \xi_5,$$
$$Y_{24} \equiv \xi_2 - \xi_3 + \xi_4, \quad Y_{23} \equiv \xi_3 + \xi_4 - \xi_5.$$

Expressing in terms of the new indices ξ_i the substitution (31) to which B corresponds, we obtain the result:

$$B \sim B' \equiv C_4 C_5 (\xi_3 \xi_5 \xi_4). \tag{33}$$

Proceeding similarly with the substitutions $B_1, \cdots, B_5$, we find that

$$B_1 \sim B_1' \equiv C_3 C_4 E_3' E_1' E_2' E_3' W' E_3' E_2' E_1'^2 E_3' C_2 C_5, \tag{34}$$

$$B_2 \sim B_2' \equiv C_2 C_3 C_4 C_5, \tag{35}$$

$$B_3 \sim B_3' \equiv C_3 C_4 B_1' C_3 C_4, \tag{36}$$

$$B_4 \sim B_4' \equiv C_4 C_5 E_3' B_1' E_3' C_4 C_5, \tag{37}$$

$$B_5 \sim B_5' \equiv C_1 C_3 C_4 C_5, \tag{38}$$

where the substitutions B_1', B_4' are in matricular notation

$$\begin{Vmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 1 \\ 1 & 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 \end{Vmatrix}, \qquad \begin{Vmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 \end{Vmatrix}.$$

We proceed next to express the generators E_1', E_2', E_3', C_1C_2, W' of the orthogonal group $O_{5,3}$ in terms of B', B_1', $\cdots$, B_5', which correspond to the generators of $H_{4,3^6}$.

$$W' = B_2' B_3' B_4' B_3', \quad E_3' = B' B_5' B' W' B_5' W' B', \quad E_2' = C_1 C_5 W' B_5' W',$$
$$C_1 C_5 = B' W' B_5' W' E_3', \quad E_1' = C_3 C_5 B_1' B'^2 B_3' B' E_3' C_1 C_2 C_1 C_5,$$
$$C_1 C_5 = B_2' B_5', \quad C_3 C_5 = E_3' C_3 C_4 E_3', \quad C_3 C_4 = B_5' C_1 C_5.$$

The corresponding Abelian substitutions have corresponding relations. The substitutions W, $W^2 \equiv W^{-1}$; E_3, E_2, E_1; C_1C_5, C_3C_4, C_3C_5, C_1C_2, C_2C_3 are found to be the following:

$$\begin{vmatrix} 0 & -1 & - & \\ 0 & 1 & 1 & 1 \\ -1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{vmatrix};$$

$$\begin{vmatrix} 1 & 0 & 0 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ -1 & 0 & 0 & -1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{vmatrix}, \quad \begin{vmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 1 \end{vmatrix};$$

$$\begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix},$$

$$\begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \quad \begin{vmatrix} -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 \end{vmatrix}.$$

These calculations have been checked in several ways.

16. THEOREM.—*If the hyperabelian group $HA_{4,3^2}$ be isomorphic with the orthogonal group $O_{6,3}$ in such a manner that the correspondences of* §15 *hold between the operators of their respective subgroups $H_{4,3}$ and $O_{5,3}$, then the substitution of the group $HA_{4,3^2}$,*

$$\bar{I} \equiv \begin{vmatrix} I & 0 & 0 & 0 \\ 0 & I^{-3} & 0 & 0 \\ 0 & 0 & I^{-1} & 0 \\ 0 & 0 & 0 & I^3 \end{vmatrix}, \tag{39}$$

in which the mark I is a suitably chosen root of the congruence

$$x^2 \equiv x + 1 \pmod 3,$$

must correspond to the substitution $C_1 C_2 (\xi_1 \xi_6)(\xi_3 \xi_4)$ which extends $O_{5,3}$ to $O_{6,3}$.

In the quotient group $HA_{4,3^2}$, every hyperabelian substitution multiplying all four indices by the same factor corresponds to the identity, viz., the powers of

$$(40) \qquad \begin{vmatrix} I^2 & 0 & 0 & 0 \\ 0 & I^2 & 0 & 0 \\ 0 & 0 & I^2 & 0 \\ 0 & 0 & 0 & I^2 \end{vmatrix} \equiv \begin{vmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{vmatrix} = 1,$$

where we have set

$$(41) \qquad I^2 \equiv i, \quad I \equiv i - 1, \quad i^2 \equiv -1 \pmod 3.$$

Then since $I^8 = 1$, one has easily

$$(42) \qquad \overline{I}^2 = \overline{C}_2 \overline{C}_3 \overline{C}_4 \overline{C}_5,$$

so that $\overline{I}$ is of period four. We readily verify the following relations:

$$(43) \qquad \begin{aligned} &\overline{I}\overline{C}_3\overline{C}_4 = \overline{C}_3\overline{C}_4\overline{I}, \quad \overline{I}\overline{C}_1\overline{C}_2 = \overline{C}_1\overline{C}_2\overline{I}^{-1}, \quad \overline{I}\overline{C}_1\overline{C}_5 = \overline{C}_1\overline{C}_5\overline{I}^{-1}, \\ &\overline{I}\overline{C}_2\overline{C}_3 = \overline{C}_3\overline{C}_5\overline{I}^{-1}, \quad \overline{E}_2\overline{E}_3\overline{I}\overline{E}_2\overline{E}_3 = \overline{I}, \quad \overline{E}_2\overline{E}_1\overline{I}\overline{E}_2\overline{E}_1 = \overline{C}_3\overline{C}_5\overline{I}^{-1}. \end{aligned}$$

Suppose that $O_{6,3}$ contains a substitution I' which combines with the generators E_1', E_2', E_3', C_1C_2, W' of $O_{5,3}$ according to the same laws by which $\overline{I}$ combines with $\overline{E}_1$, $\overline{E}_2$, $\overline{E}_3$, $\overline{C}_1\overline{C}_2$, $\overline{W}$ of $H_{4,3}$. Assume for I' the most general form possible, viz.,

$$I' \equiv \pm \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{16} \\ a_{21} & a_{22} & \cdots & a_{26} \\ \cdot & \cdot & \cdot & \cdot \\ a_{61} & a_{62} & \cdots & a_{66} \end{vmatrix},$$

the simultaneous change of sign of every coefficient leaving I' unchanged.

In virtue of the relation corresponding to (42), viz.,

$$I'^{-1} = I' C_2 C_3 C_4 C_5,$$

we find that

$$\begin{vmatrix} a_{11} & a_{21} & \cdots & a_{61} \\ a_{12} & a_{22} & \cdots & a_{62} \\ \cdot & \cdot & \cdot & \cdot \\ a_{15} & a_{25} & \cdots & a_{65} \\ a_{16} & a_{26} & \cdots & a_{66} \end{vmatrix} = \pm \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{16} \\ -a_{21} & -a_{22} & \cdots & -a_{26} \\ \cdot & \cdot & \cdot & \cdot \\ -a_{51} & -a_{52} & \cdots & -a_{56} \\ a_{61} & a_{62} & \cdots & a_{66} \end{vmatrix}.$$

According to the sign $\pm$, we find for I'_+, I'_- the respective values:

$$\pm \begin{vmatrix} a_{11} & 0 & 0 & 0 & 0 & a_{16} \\ 0 & 0 & a_{23} & a_{24} & a_{25} & 0 \\ 0 & -a_{23} & 0 & a_{34} & a_{35} & 0 \\ 0 & -a_{24} & -a_{34} & 0 & a_{45} & 0 \\ 0 & -a_{25} & -a_{35} & -a_{45} & 0 & 0 \\ a_{16} & 0 & 0 & 0 & 0 & a_{66} \end{vmatrix}, \quad \pm \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & a_{16} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} & 0 \\ 0 & a_{23} & a_{33} & a_{34} & a_{35} & 0 \\ 0 & a_{24} & a_{34} & a_{44} & a_{45} & 0 \\ 0 & a_{25} & a_{35} & a_{45} & a_{55} & 0 \\ -a_{16} & 0 & 0 & 0 & 0 & 0 \end{vmatrix}.$$

By the relation corresponding to the first relation (43), I' must be commutative with C_3C_4. We find in consequence for I'_+, I'_- the values:

$$\pm \begin{vmatrix} a_{11} & 0 & 0 & 0 & 0 & a_{16} \\ 0 & 0 & 0 & 0 & a_{25} & 0 \\ 0 & 0 & 0 & a_{34} & 0 & 0 \\ 0 & 0 & -a_{34} & 0 & 0 & 0 \\ 0 & -a_{25} & 0 & 0 & 0 & 0 \\ a_{16} & 0 & 0 & 0 & 0 & a_{66} \end{vmatrix}, \quad \pm \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & a_{16} \\ 0 & a_{22} & 0 & 0 & a_{25} & 0 \\ 0 & 0 & a_{33} & a_{34} & 0 & 0 \\ 0 & 0 & a_{34} & a_{44} & 0 & 0 \\ 0 & a_{25} & 0 & 0 & a_{55} & 0 \\ -a_{16} & 0 & 0 & 0 & 0 & 0 \end{vmatrix}.$$

The resulting substitution I'_+ does not satisfy the relation corresponding to (43),

$$I'C_1C_2 = C_1C_2I'^{-1},$$

and is therefore excluded. And I'_- satisfies it if and only if $a_{25} = 0$.

The relation $I'C_1C_5 = C_1C_5I'^{-1}$ is then satisfied by I'_-. But the relation:

$$I'C_2C_3 = C_3C_5I'^{-1},$$

requires that $a_{33} = a_{44} = 0$ in I'_-. The relations:

$$E'_2E'_3I'_-E'_2E'_3 = I'_-, \quad E'_2E'_1I'_-E'_2E'_1 = C_3C_5I'^{-1}_-,$$

require in succession that $a_{34} = a_{55}$, $a_{34} = -a_{22}$. Evidently $a_{16} \not\equiv 0$, so that $a_{16} \equiv \pm 1 \pmod 3$. By suitable choice of the sign $\pm$ in front of the matrix for I'_-, we may take $a_{16} = +1$. Setting, for brevity, $a_{34} = \gamma$, we have

$$I'_- \equiv \pm \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, according as $\gamma = \pm 1$, we have

$$I_-'^{\pm 1} = C_1 C_2 (\xi_1 \xi_6)(\xi_3 \xi_4).$$

But, by (39),

$$I^{-1} \equiv I^3 = \begin{pmatrix} I^3 & 0 & 0 & 0 \\ 0 & I^{-9} & 0 & 0 \\ 0 & 0 & I^{-3} & 0 \\ 0 & 0 & 0 & I^9 \end{pmatrix}$$

is derived from I by replacing I by I^3. But I is defined as one root of the irreducible congruence:

$$I^2 \equiv I + 1 \pmod 3,$$

whose second root is I^3. Hence, by a proper choice of notation for the root I, we may set

$$I'_- = C_1 C_2 (\xi_1 \xi_6)(\xi_3 \xi_4) = C_1 C_2 F' E_1'^2 E_2'.$$

Hence

$$F' = C_1 C_2 I'_- E'_2 E'_1.$$

It follows that to F' must correspond the hyperabelian substitution

$$(44) \qquad \overline{F} \equiv \overline{C}_1 \overline{C}_2 I \overline{E}_2 \overline{E}_1 \equiv \begin{pmatrix} 0 & 0 & -I & 0 \\ 0 & 0 & -I & I \\ I^3 & 0 & 0 & 0 \\ I^3 & -I^3 & 0 & 0 \end{pmatrix}.$$

17. THEOREM.—*The hyperabelian group* $HA_{4,3^2}$ *is holoedrically isomorphic* with the abstract group* Γ.

From the simple isomorphism of $H_{4,3}$ with G, it follows that the operators $E_1, E_2, E_3, B_1 \equiv C_1C_2, W$ satisfy the generational relations (1), ⋯, (6) of G, a result capable of verification by direct calculation. It therefore remains only to prove that the operators $F, E_1, E_2, B_1 \equiv C_1C_2, W$ satisfy the generational relations (12), ⋯, (18). This result may be verified by simple calculations. We note the auxiliary formulæ:

$$C_4C_5 = \begin{vmatrix} -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix}, \quad E_3WE_3 = \begin{vmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{vmatrix},$$

$$W^2C_1C_2C_4C_5 = \begin{vmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ -1 & -1 & 0 & 0 \end{vmatrix}, \quad V = \begin{vmatrix} 1 & 0 & -i & 0 \\ 1 & -1 & -i & i \\ i & 0 & -1 & 0 \\ i & -i & -1 & 1 \end{vmatrix}.$$

We have therefore proved that the simple groups $HA_{4,3^2}$ and $O_{6,3}$ are holoedrically isomorphic.

UNIVERSITY OF TEXAS,
January 12, 1900.

*Addition: May 5, 1900. In the May number of the Bulletin of the Society the writer establishes the holoedric isomorphism of O_{6,p^n} and $HA_{4,p^{2n}}$ for any p^n of the form $4l-1$. As the method there used consists in the transformation of the defining invariant of the former group into that of the second compound of the latter group, it gives no direct knowledge of the correspondences of the generators of the isomorphic groups. For $p^n = 3$, §§ 15-16 of the present paper enable us to pass readily from an arbitrary substitution of either group to the corresponding substitution of the other.